Brief solutions to Midterm 1

Mar 24, 2022.

1. (12 pts) Are the integrals (a): $\int_0^1 \frac{1}{\sqrt{x+x^3}}\,dx$ and (b): $\int_1^\infty \tan(\frac{1}{x})\,dx$ convergent? Explain.

   Answer:

   (a): Since $\lim_{x\to 0} \frac{\frac{1}{\sqrt{x+x^3}}}{\frac{1}{\sqrt{x}}} = 1$ and $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges **(2 pts)**, therefore $\int_0^1 \frac{1}{\sqrt{x+x^3}}\,dx$ converges by Limit Comparison Test (**4pts**).

   (b): Since $\lim_{x\to\infty} \frac{\tan(\frac{1}{x})}{\frac{1}{x}} = 1$ and $\int_1^\infty \frac{1}{x}\,dx$ diverges **(2 pts)**, therefore $\int_1^\infty \tan(\frac{1}{x})\,dx$ diverges by Limit Comparison Test (**4pts**).

2. (12 pts) Evaluate $\lim_{n\to\infty} \frac{\log\left(\sum_{k=n}^{\infty} k^{-3}\right)}{\log n}$. Give details.

   Hint: If the limit is p, this means that $\sum_{k=n}^{\infty} k^{-3}$ is approximately n^p. Find p and prove it. Recall the proof of one of the convergence tests.

   Answer:

$$\because\ f(k) = k^{-3}\ \geq 0 \searrow \text{ and } \int_1^\infty \frac{1}{x^3}\,dx \text{ converges. } \therefore \text{ Converges.}$$

$\because$

$$\frac{1}{2n^2} = \int_n^\infty \frac{1}{x^3}\,dx \leq \sum_{k=n}^{\infty} \frac{1}{k^3} \leq \int_{n-1}^\infty \frac{1}{x^3}\,dx = \frac{1}{2(n-1)^2}$$

$$\Rightarrow \frac{-(\log 2 + 2\log n)}{\log n} \leq \frac{\log\left(\sum_{k=n}^{\infty} k^{-3}\right)}{\log n} \leq \frac{-(\log 2 + 2\log(n-1))}{\log n}$$

Let $n \to \infty$, we see that the answer = -2.

3. (12 pts) find $\sum_{n=1}^{\infty} nx^n$ and $\sum_{n=1}^{\infty} n^2x^n$ on $|x| < 1$ using computational rules for power series (multiplication, differentiation, integration, etc.).

Answer:

$$
\begin{aligned}
& 1 + x + x^2 + \ldots = \frac{1}{1-x} \ \textbf{(2 pts)} \\
\Rightarrow \quad & x + 2x^2 + 3x^3 + \ldots = x\left(\frac{1}{1-x}\right)' = \frac{x}{(1-x)^2} \ \textbf{(5 pts)} \\
\Rightarrow \quad & x + 2^2x^2 + 3^2x^3 + \ldots = x\left(\frac{x}{(1-x)^2}\right)' = \frac{x(1+x)}{(1-x)^3} \ \textbf{(5 pts)}.
\end{aligned}
$$

4. (12 pts)

(a) Show that the series $1 - \frac{1}{2 \cdot 1!} + \frac{1}{4 \cdot 2!} - \cdots + (-1)^n \frac{1}{2^n \cdot n!} + \cdots$ converges *absolutely*.

(b) Find the sum of the series in (a). Prove your answer (that is, the equality holds).

Answer:

(a) Ratio test:

$$\lim_{n\to\infty} \frac{\frac{1}{2^{n+1}(n+1)!}}{\frac{1}{2^n n!}} = 0 < 1, \quad \text{convergent. } \textbf{(6 pts)}$$

(b) Sum $= e^{-1/2}$. **(2 pts)** Since

$$1 - \frac{1}{2 \cdot 1!} + \frac{1}{4 \cdot 2!} - \cdots + (-1)^n \frac{1}{2^n \cdot n!} + \cdots = T_{e^x,0}(x = -\frac{1}{2})$$

$$T_{e^x,0}(-\frac{1}{2}) = e^{-\frac{1}{2}} \text{ if and only if } \lim_{n\to\infty} R_n(-\frac{1}{2}) = 0.$$

$$|R_n(-\frac{1}{2})| \le \frac{e^{c_{n+1}}}{(n+1)!} \frac{1}{2^{n+1}} \le \frac{1}{(n+1)!} \frac{1}{2^{n+1}} \to 0 \text{ where } c_{n+1} \in (-\frac{1}{2}, 0) \ . \ \textbf{(4 pts)}$$

5. (12 pts) Find $T_{\sin^{-1},0}(x)$ and its radius of convergence.

Answer:

$$\sin^{-1} x = \int_0^x (1-t^2)^{\frac{-1}{2}} dt =$$

$$= \int_0^x (1 - \frac{1}{2}(-t^2) + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}(-t^2)^2 - \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}(-t^2)^3 + \cdots + + (-1)^n \frac{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2}}{n!}(-t^2)^n + \cdots)dt$$

$$= x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7} + \cdots + \frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}\cdot\frac{x^{2n+1}}{2n+1} + \cdots \quad \textbf{(8 pts)}$$

Radius of convergence = 1 by ratio test (**4 pts**).

6. (12 pts) Give an approximation of $\int_0^{\frac{1}{2}} \sin(x^2)\, dx$ to within 10^{-8}. Give the formula of the approximation, but need not find the numerical value. Explain why the error is less than 10^{-8}.

Answer:

$$\sin x^2 = x^2 - \frac{1}{3!}(x^2)^3 + \frac{1}{5!}(x^2)^5 - \frac{1}{7!}(x^2)^7 + \cdots = \sum_{k=0}^{\infty}(-1)^k \frac{(x^2)^{2k+1}}{(2k+1)!}$$

From the error estimate for alternating series:

$$\left|\sin x^2 - \sum_{k=0}^{n}(-1)^k \frac{(x^2)^{2k+1}}{(2k+1)!}\right| \leq \left|\frac{(x^2)^{2n+3}}{(2n+3)!}\right|$$

$$\text{error} = \left|\int_0^{\frac{1}{2}} \sin x^2\, dx - \sum_{k=0}^{n}\int_0^{\frac{1}{2}}(-1)^k \frac{(x^2)^{2k+1}}{(2k+1)!}\, dx\right| \leq \left|\int_0^{\frac{1}{2}} \frac{(x^2)^{2n+3}}{(2n+3)!}\, dx\right| \quad (\textbf{4pts})$$

We want

$$\int_0^{\frac{1}{2}} \frac{(x^2)^{2n+3}}{(2n+3)!}\, dx = \frac{1}{(4n+7)2^{4n+7}(2n+3)!} < 10^{-8}$$

Any $n \geq 2$, will do since for $n = 2$, $(4n+7)2^{4n+7}(2n+3)! = 15 \cdot 2^{15} \cdot 7! > 10^8$ (**4pts**).

Answer:

$$\text{Approximation} = \sum_{k=0}^{2}(-1)^k \frac{1}{(4k+3)\cdot 2^{4k+3}\cdot(2k+1)!} \ (\textbf{4 pts}).$$

7. (6 pts) Evaluate $\lim_{x\to 0} \frac{\frac{\sin x}{x} - \cos x}{x^2}$.

Answer:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$

Answer:

$$\lim_{x\to 0} \frac{\frac{\sin x}{x} - \cos x}{x^2} = \lim_{x\to 0} \frac{(1 - \frac{x^2}{3!} + \cdots) - (1 - \frac{x^2}{2!} + \cdots)}{x^2} = \frac{1}{3} \quad (\textbf{6pts})$$

8. (6 pts) Let $f(x) = \frac{\sin x}{\cos 2x}$. Find $T_{f,0}(x)$ upto x^5 term.

Answer:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \quad \cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots,$$

On $|x| < \dfrac{\pi}{4}$ where $\cos 2x \neq 0$, we can use long division to get

$$f(x) = \frac{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots}{1 - 2x^2 + \frac{2}{3}x^4 - \cdots} = x + \frac{11}{6}x^3 + \frac{361}{120}x^5 + \cdots (\mathbf{2+2+2pts}),$$

Since f has a power series representation on an open interval, therefore $f(x) = T_{f,0}(x)$ (See problem 9(a)).

9. (4+4+8+8 pts) True or False? Prove it if true, give a counter example if false.

(a) If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on $|x| < 1$, then $a_n = \dfrac{f^{(n)}(0)}{n!}$.

(b) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} n a_n$ converges.

(c) If $g(x) = f(0) + \sum_{n=1}^{\infty} \dfrac{f^{(n)}(0)}{n!} x^n$ on $|x| < 1$, then $f(x) = g(x)$ on $|x| < 1$.

(d) If $\sum_{n=0}^{\infty} a_n x^n$ converges on $|x| < 1$, then $\sum_{n=0}^{\infty} \sqrt{n} a_n x^n$ also converges on $|x| < 1$.

Answer:

(a) True **(1 pts)**. By term by term differentiation Theorem **(3 pts)**.

(b) False **(1 pts)**. $\sum_n \dfrac{(-1)^n}{n}$ converges, but $\sum_n (-1)^n$ diverges **(3 pts)**.

(c) False **(2 pts)**. Take $f(x) = e^{-1/x^2}$ for $x \neq 0$, and $f(x) = 0$ for $x = 0$. Then $f^{(n)}(0) = 0$, $\forall\ n \geq 0$. Therefore $f(0) + \sum_{n=1}^{\infty} \dfrac{f^{(n)}(0)}{n!} x^n = 0 \neq f(x)$, if $x \neq 0$ **(correct example: 2 pts, correct explanation: 4 pts)**.

(d) True **(4 pts)**.

Proof 1: Given $|x| < 1$, take y with $|x| < |y| < 1$, we have $\sum_n a_n y^n$ converges absolutely. Moreover, $|\sqrt{n} a_n x^n| \leq |a_n y^n|$ for n large enough since $\sqrt{n} \left|\dfrac{x}{y}\right|^n \to 0$ as $n \to \infty$. From direct comparison test, $\sum_n \sqrt{n} a_n x^n$ also converges absolutely.

Proof 2: Firstly, $\sum_n a_n x^n$ converges absolutely. From term by term differentiation Theorem, $\sum_n n a_n x^n$ converges absolutely. Since $|\sqrt{n} a_n x^n| \leq |n a_n x^n|$, from direct comparison test, $\sum_n \sqrt{n} a_n x^n$ also converges absolutely.

(4 pts).

Remark: We know that the radius of convergence $R \geq 1$. But we can conclude that $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{R}$ only if $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ exist, which may not be true. Similarly for $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. See the example in page 9 of lecture 06.