Brief solutions to selected problems in homework week 16

1. Section 16.5. problem 56:

56 (b) $r(x,\theta) = (x, f(x)\cos\theta, f(x)\sin\theta)$

$a \le x \le b$, $0 \le \theta \le 2\pi$

$r_x = (1, f'\cos\theta, f'\sin\theta)$, $r_\theta = (0, -f\sin\theta, f\cos\theta)$

$r_x \times r_\theta = (f'f, -f\cos\theta, -f\sin\theta)$

$$\Rightarrow \iint d\sigma = \int_0^{2\pi}\int_a^b |(f'f, -f\cos\theta, -f\sin\theta)|\, dx\, d\theta$$

$$= \int_0^{2\pi}\int_a^b f(x)\sqrt{1+f'(x)^2}\, dx\, d\theta$$

$$= 2\pi\int_a^b f(x)\sqrt{1+f'(x)}\, dx\, d\theta$$

2. Section 16.7. problem 7:

Sec 16.7

6. $\oint F\cdot d\vec{r} = \iint_S (\nabla\times F)\cdot n\, d\sigma$

$n = (0,0,1) \Rightarrow (\nabla\times F)\cdot n = 3x^2y^2$

所求 $= \iint_S 3x^2y^2\, d\sigma$

$x = r\cos\theta$
$y = r\sin\theta$
$d\sigma = r\,dr\,d\theta$

$$= \int_0^2\int_0^{2\pi} 3r^2\cos^2\theta\, r^2\sin^2\theta\, r\, d\theta\, dr$$

$= 8\pi$

3. Section 16.7. problem 19:

19. $F = (y, -x, x^2)$, $r(t) = (2\cos t, 2\sin t, 3-2\cos^3 t)$, $0 \le t \le 2\pi$

$$\oint F \cdot d\vec{r} = \int_0^{2\pi} \begin{pmatrix} y \\ -x \\ x^2 \end{pmatrix} \begin{pmatrix} -2\sin t \\ 2\cos t \\ +6\cos^2 t \sin t \end{pmatrix} dt$$

$$= \int_0^{2\pi} -4 + 24\cos^4 t \sin t \, dt = -8\pi$$

$$\iint_S \nabla \times F \cdot n \, d\sigma = \iint \nabla \times F \cdot n \, d\sigma + \iint \nabla \times F \cdot n \, d\sigma$$

$$\nabla \times F = (0, -2x, -2)$$

$$\iint \nabla \times F \cdot n \, d\sigma = \iint \begin{pmatrix} 0 \\ -2x \\ -2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} d\sigma = \iint_{x^2+y^2 \le 4} -2 \, d\sigma = -8\pi$$

$$\iint \nabla \times F \cdot n \, d\sigma = \iint \begin{pmatrix} 0 \\ -2x \\ -2 \end{pmatrix} \begin{pmatrix} -x \\ -y \\ 0 \end{pmatrix} \frac{d\sigma}{\sqrt{x^2+y^2}}$$

$r(\theta, z) = (2\cos\theta, 2\sin\theta, z)$

$0 \le \theta \le 2\pi$, $0 \le z \le 2 - \cos^3\theta$

$\Rightarrow |r_\theta \times r_z| = 2$

$$\Rightarrow \iint \frac{2xy}{\sqrt{x^2+y^2}} d\sigma = \int_0^{2\pi} \int_0^{3-\cos^3\theta} \frac{2(2\cos\theta)(2\sin\theta)}{2} 2 \, dz \, d\theta$$

$$= 0.$$

Thus $\oint F d\vec{r} = \iint_S \nabla \times F \cdot n \, d\sigma$