# Brief solutions to selected problems in homework week 15

1. Problem 1:

**1. (16.3)**

$$F = \frac{x}{\sqrt{x^2+y^2}}\,i + \frac{y}{\sqrt{x^2+y^2}}\,j + 0\,k$$

$$G = \frac{-y}{x^2+y^2}\,i + \frac{x}{x^2+y^2}\,j + 0\,k$$

a $M_1 = \frac{x}{\sqrt{x^2+y^2}}$, $N_1 = \frac{y}{\sqrt{x^2+y^2}}$, $P_1 = 0$

$\frac{\partial P_1}{\partial y} = 0 = \frac{\partial N_1}{\partial z}$, $\frac{\partial P_1}{\partial x} = 0 = \frac{\partial M_1}{\partial z}$, $\frac{\partial M_1}{\partial y} = -\frac{xy}{(x^2+y^2)^{\frac{3}{2}}} = \frac{\partial N_1}{\partial x}$

$\Rightarrow$ F satisfies the component test

$M_2 = \frac{-y}{x^2+y^2}$, $N_2 = \frac{x}{x^2+y^2}$, $P_2 = 0$

$\frac{\partial P_2}{\partial y} = 0 = \frac{\partial N_2}{\partial z}$, $\frac{\partial P_2}{\partial x} = 0 = \frac{\partial M_2}{\partial z}$, $\frac{\partial M_2}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial N_2}{\partial x}$

$\Rightarrow$ G satisfies the component test b $\nabla f = F$

$$\frac{\partial f}{\partial x} = M_1 , \frac{\partial f}{\partial y} = N_1 , \frac{\partial f}{\partial z} = P_1$$

$$f(x,y,z) = \sqrt{x^2+y^2} + g(y,z)$$

$$\frac{y}{\sqrt{x^2+y^2}} + \frac{\partial g}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \Rightarrow \frac{\partial g}{\partial y} = 0$$

$$\Rightarrow f(x,y,z) = \sqrt{x^2+y^2} + h(z)$$

$$0 + \frac{\partial h}{\partial z} = 0 \Rightarrow \frac{\partial h}{\partial z} = 0 , \quad h(z) = \overset{C}{\cancel{z+C}}$$

$$\Rightarrow f(x,y,z) = \sqrt{x^2+y^2} \cancel{+z} + C$$

C $r(t) = (\cos t)\, i + (\sin t)\, j\ ,\ 0 \le t \le 2\pi$

$$G = \frac{-y}{x^2+y^2}\, i + \frac{x}{x^2+y^2}\, j$$

$$= \frac{-\sin t}{\sin^2 t + \cos^2 t}\, i + \frac{\cos t}{\sin^2 t + \cos^2 t}\, j$$

$$= (-\sin t)\, i + (\cos t)\, j$$

$$\frac{dr}{dt} = (-\sin t)\, i + (\cos t)\, j$$

$$\oint G \cdot dr = \oint_C G \cdot \frac{dr}{dt}\, dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt$$

$$= 2\pi \neq 0$$

$\because \oint G \cdot dr \neq 0$

$\therefore$ G isn't conservative by Thm 3.

(d) Find if there exists $h$, a differential function, such that $H = \nabla h$

Problem 1 (d): It is easier to explain the idea if we restrict problem 1 in the plane:

Let $\boldsymbol{F} = \dfrac{x}{\sqrt{x^2+y^2}}\,\boldsymbol{i} + \dfrac{y}{\sqrt{x^2+y^2}}\,\boldsymbol{j}$ and $\boldsymbol{G} = \dfrac{-y}{x^2+y^2}\,\boldsymbol{i} + \dfrac{x}{x^2+y^2}\,\boldsymbol{j}$.

(a) Show that both $\boldsymbol{F}$ and $\boldsymbol{G}$ satisfy the component test.

(b) The natural domain of both $\boldsymbol{F}$ and $\boldsymbol{G}$ is $\{(x, y),\ x^2 + y^2 \neq 0\}$ (that is where $\boldsymbol{F}$ and $\boldsymbol{G}$ are defined). Show that $\boldsymbol{F}$ is conservative in this domain by finding its potential function.

(c) Show that $\boldsymbol{G}$ is NOT conservative in this domain (see Example 5 on p990).

(d) If given another $\boldsymbol{H}$ satisfying the component test in this domain, how do you determine whether $\boldsymbol{H}$ is conservative?

**Ans**: It is clear that answers to (a), (b), (c) remain unchanged.

For (d): Suppose $\boldsymbol{H}$ satisfies the component test in $\{(x,y),\ x^2+y^2\neq 0\}$. Let $C$ be any simple closed curve, and $\Omega$ be the inside of $C$.

(a) If $(0,0)\notin\Omega$.

In this case, $\Omega$ is simply connected. We can apply the 2D version of 'Component Test for Conservative Field" statement on page 988, to conclude that ($\boldsymbol{H}$ is conservative, and therefore)

$$\oint_C \boldsymbol{H}\cdot\boldsymbol{T}\,ds = 0 \tag{1}$$

(b) If $(0,0)\in\Omega$.

As explained in Lecture 27, page 7 (see the figure there), using the same argument as in case (a) above, we have (replace $\Omega$ by $\mathcal{R}_{\epsilon,1}$ and $\mathcal{R}_{\epsilon,2}$),

$$\oint_{\partial\mathcal{R}_{\epsilon,1}} \boldsymbol{H}\cdot\boldsymbol{T}\,ds = 0, \qquad \oint_{\partial\mathcal{R}_{\epsilon,2}} \boldsymbol{H}\cdot\boldsymbol{T}\,ds = 0$$

As a result, we have

$$\oint_C \boldsymbol{H}\cdot\boldsymbol{T}\,ds = \oint_{C_\epsilon} \boldsymbol{H}\cdot\boldsymbol{T}\,ds \tag{2}$$

where $C_\epsilon = \{(x,y),\ x^2+y^2=\epsilon^2\}$. Moreover, it is clear that the line integral in (2) is independent of $\epsilon>0$.

We conclude from the above analysis that,

(a) If $\oint_{C_\epsilon} \boldsymbol{H}\cdot\boldsymbol{T}\,ds \neq 0$, then from Theorem 3 (loop property), $\boldsymbol{H}$ is not conservative.

(b) If $\oint_{C_\epsilon} \boldsymbol{H}\cdot\boldsymbol{T}\,ds = 0$, then we conclude from (1), (2) that

$$\oint_C \boldsymbol{H}\cdot\boldsymbol{T}\,ds = 0 \tag{3}$$

for every simple closed curve $C$.

If $C$ is closed but not simple (i.e. $C$ intersects itself), we can always decompose $C$ into several simple closed curves (break up at the intersection points and reconnect), it follows that (3) remains valid even if $C$ is not simple closed.

In summary, we have the following conclusion:

$$\boldsymbol{H}\ \text{ is conservative} \iff \oint_C \boldsymbol{H}\cdot\boldsymbol{T}\,ds = 0 \text{ for any closed curve } C \iff \oint_{C_\epsilon} \boldsymbol{H}\cdot\boldsymbol{T}\,ds = 0 \tag{4}$$

The conclusion (4) remains valid in 3D. The argument is similar, with the following replacement of key words:

2D: If $C$ is simple closed and $(0,0) \notin \Omega$. (3D: If $C$ does not circle around the $z$-axis).

2D: If $C$ is simple closed and $(0,0) \in \Omega$. (3D: If $C$ circles around the $z$-axis once).

2D: $C_\epsilon = \{(x,y),\ x^2+y^2=\epsilon^2\}$. (3D: $C_\epsilon = \{(x,y,z=0),\ x^2+y^2=\epsilon^2\}$).

2D: If $C$ is not simple closed. (3D: If $C$ circles around the $z$-axis more than once).

2. Problem 2:

2. Let $\vec{F} = \frac{1}{\sqrt{x^2+y^2+z^2}}(x, y, z)$.

(a) What is the natural domain $D_F$ of $\vec{F}$?

(b) Show that $\vec{F}$ satisfies component test in $D_F$.

(c) Is $D_F$ simply connected?

(d) Is $\vec{F}$ conservative in this domain?

(a) $D_F = \{(x,y,z) \mid x^2+y^2+z^2 > 0\} = \mathbb{R}^3 \setminus \{(0,0,0)\}$

(b)

$$\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) = -xy(x^2+y^2+z^2)^{-\frac{3}{2}} = \frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\frac{\partial}{\partial z}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) = -xz(x^2+y^2+z^2)^{-\frac{3}{2}} = \frac{\partial}{\partial x}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\frac{\partial}{\partial z}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) = -yz(x^2+y^2+z^2)^{-\frac{3}{2}} = \frac{\partial}{\partial y}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$\therefore \vec{F}$ satisfies component test in $D_F$

(c)

$D_F$ is simply connected

(d) By (b), $\vec{F}$ satisfies component test in $D_F$.

Also, $D_F$ is simply connected.

$\therefore \vec{F}$ is conservative in $D_F$

Problem 2(d):

**Method 2**: By observation (or whatever methods), we know that $\boldsymbol{F} = \nabla\sqrt{x^2+y^2+z^2}$, therefore $\boldsymbol{F}$ is conservative.

3. Section 16.4. problem 10:

16.4.

#10.

$\vec{F} = (\underbrace{x+3y}_{M}, \underbrace{2x-y}_{N})$

$C: x^2+2y^2=2$

$r(t) = (\cancel{2\cos t}\ \sqrt{2}\cos t, \sin t) \quad 2\pi \ge t \ge 0$

use Green's Thm. $\oint \vec{F}(r(t)) \cdot r'(t)\, dt = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dxdy$ (flow)

$\oint M dy - N dx = \iint_R \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\, dxdy$ (flux)

circulation $= \iint_R (2-3)\, dxdy$

$= \int_{-1}^{1}\int_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}} (-1)\, dx\, dy$

$= \int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{\frac{2-x^2}{2}}}^{\sqrt{\frac{2-x^2}{2}}} (-1)\, dy\, dx$

$= -\frac{2}{\sqrt{2}}\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{2-x^2}\, dx$ ⇒ 圓上半 $(r=\sqrt{2})$

$= -\sqrt{2}\pi$

flux $= \iint_R \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \iint_R 0\, dxdy = 0$

4. Section 16.4. problem 38:

38.

$M = \frac{1}{4}x^2y + \frac{1}{3}y^3$

$N = x$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{1}{4}x^2 + y^2$

$\frac{\partial N}{\partial x} = 1$

$\Rightarrow \text{Curl} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - \left(\frac{1}{4}x^2 + y^2\right) \geq 0$

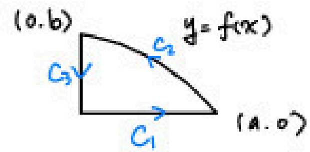
$\Rightarrow \text{work} = \oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(1 - \frac{1}{4}x^2 - y^2\right) dx\,dy$

has maximum value on $R = \{(x,y) \mid \text{curl}\,\vec{F} \geq 0\}$

or region enclosed by $\left(1 = \frac{1}{4}x^2 + y^2\right)$

5. Problem 4:

4. $F = (M(x,y),\ \underbrace{0}_{N=0})$ on $R$




$C_1: \begin{cases} x = t \\ y = 0 \end{cases} \quad 0 \le t \le a$

$C_2: \begin{cases} x = a-t \\ y = f(a-t) \end{cases} \quad 0 \le t \le a$

$C_3: \begin{cases} \boxed{x=0} \\ y = b-t \end{cases} \quad 0 \le t \le b$

Tangential form :

$$\iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

$$= \int_0^a \int_0^{f(x)} -\frac{\partial M}{\partial y}\, dy\,dx$$

$$= \int_0^a (-M(x,f(x)) + M(x,0))\, dx$$

$$= -\int_0^a M(x,f(x))\,dx + \int_0^a M(x,0)\,dx$$

$$\oint_C M\,dx - N\,dy$$

$$= \oint_{C_1} M\,dx + \oint_{C_2} M\,dx + \cancelto{0}{\oint_{C_3} M\,dx}$$

$$= \int_0^a M(t,0)\,dt + \int_0^a M(a-t,\ f(a-t))(-dt)$$

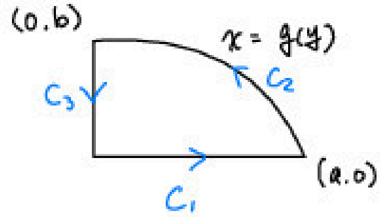
let $x = a-t$
$dx = -dt$

$$= \int_0^a M(t,0)\,dt + \int_a^0 M(x,f(x))\,dx$$

$$= \int_0^a M(x,0)\,dx - \int_0^a M(x,f(x))\,dx$$

$$\therefore \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy = \oint_C M\,dx + N\,dy$$

Normal form:




$$C_1:\begin{cases} x=t \\ \boxed{y=0} \end{cases} \quad 0\le t\le a$$

$$C_2:\begin{cases} x=g(t) \\ y=t \end{cases} \quad 0\le t\le b$$

$$C_3:\begin{cases} x=0 \\ y=b-t \end{cases} \quad 0\le t\le b$$

$$\iint_R \left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right) dx\,dy$$

$$= \int_0^b \int_0^{g(y)} \frac{\partial M}{\partial x}\, dx\, dy$$

$$= \int_0^b \big(M(g(y),y) - M(0,y)\big)\, dy$$

$$= \int_0^b M(g(y),y)\, dy - \int_0^b M(0,y)\, dy$$

$$\oint_C M\,dy - N\,dx$$

$$= \oint_{C_1} \cancelto{0}{M\,dy} + \oint_{C_2} M\,dy + \oint_{C_3} M\,dy$$

$$= \int_0^b M(g(t),t)\, dt + \int_0^b M(0,b-t)(-dt)$$

let $y = b-t$
$dy = -dt$

$$= \int_0^b M(g(t),t)\, dt + \int_b^0 M(0,y)\, dy$$

$$= \int_0^b M(g(y),y)\, dt - \int_0^b M(0,y)\, dy$$

$$\therefore \iint_R \left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right) dx\,dy = \oint_C M\,dy - N\,dx$$