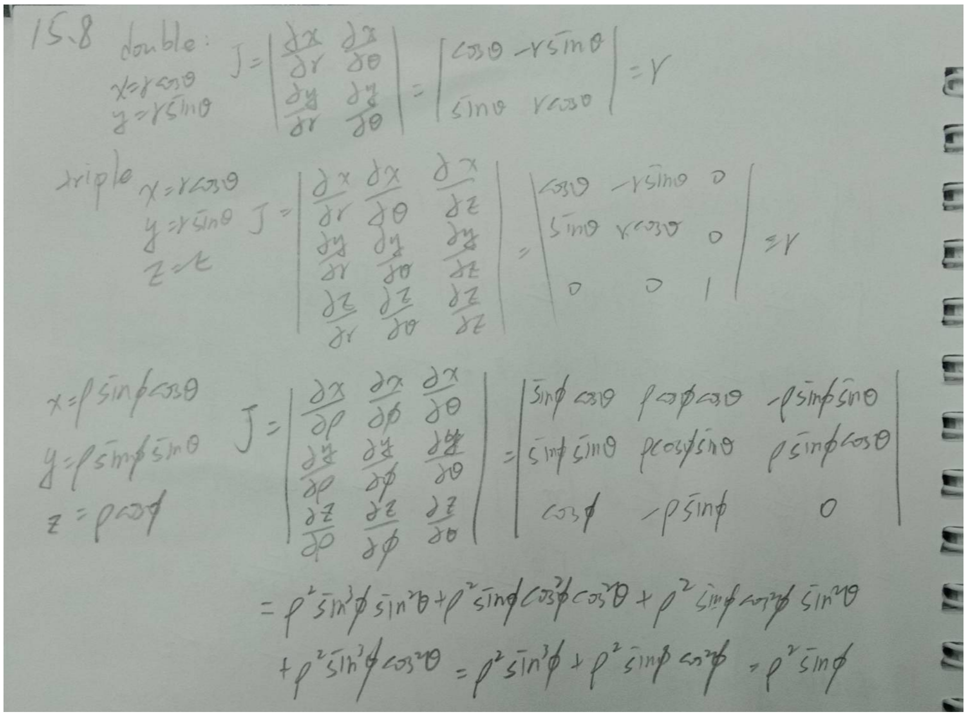
Brief solutions to selected problems in homework week 13

1. Problem 3:



15.8 double: $x = r\cos\theta$, $y = r\sin\theta$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

triple: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

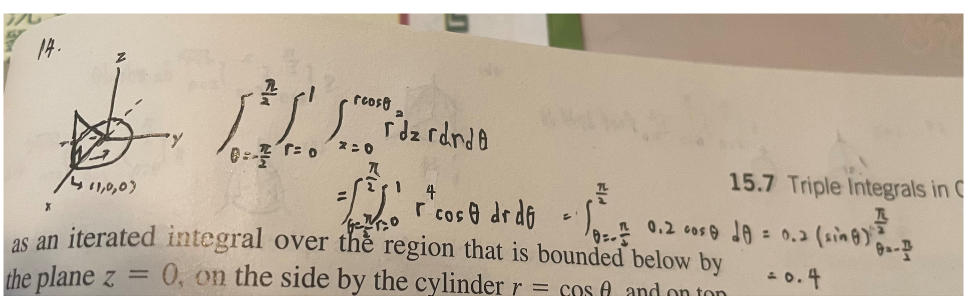
$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \rho^2\sin^3\phi\sin^2\theta + \rho^2\sin\phi\cos^2\phi\cos^2\theta + \rho^2\sin\phi\cos^2\phi\sin^2\theta + \rho^2\sin^3\phi\cos^2\theta = \rho^2\sin^3\phi + \rho^2\sin\phi\cos^2\phi = \rho^2\sin\phi$$

2. Section 15.7, problem 14:



14.

$$\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{r=0}^{1}\int_{z=0}^{r\cos\theta} r^2\,dz\,r\,dr\,d\theta = \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{r=0}^{1} r^4\cos\theta\,dr\,d\theta = \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} 0.2\cos\theta\,d\theta = 0.2(\sin\theta)\Big|_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0.4$$

3. Section 15.7, problem 31:

31

31. Let D be the region in Exercise 11. Set up the triple integrals in spherical coordinates that give the volume of D using the following orders of integration.

a. $d\rho\, d\phi\, d\theta$ **b.** $d\phi\, d\rho\, d\theta$

11. Let D be the region bounded below by the plane $z = 0$, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$. Set up the triple integrals in cylindrical coordinates that give the volume of D using the following orders of integration.

a. $dz\, dr\, d\theta$ **b.** $dr\, dz\, d\theta$ **c.** $d\theta\, dz\, dr$

a. $\boxed{x^2+y^2=1}$

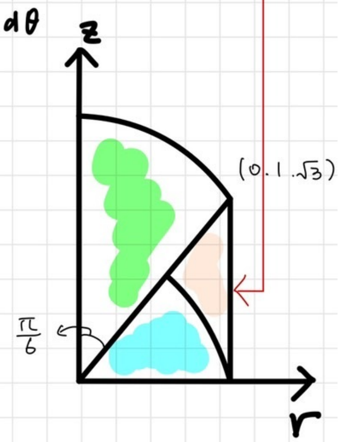
$\Rightarrow \rho^2 \sin^2\phi = 1$

$\rho \sin\phi = 1$

$\rho = \csc\phi$

$$\Rightarrow \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{2} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta + \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^{\csc\phi} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

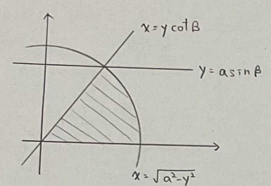
b. $$\int_0^{2\pi}\int_1^{2}\int_{\cancel{0}\ \frac{\pi}{6}}^{\sin^{-1}(\frac{1}{\rho})} \rho^2 \sin\phi \, d\phi\, d\rho\, d\theta + \int_0^{2\pi}\int_0^{2}\int_0^{\frac{\pi}{6}} \rho^2 \sin\phi \, d\phi\, d\rho\, d\theta + \int_0^{2\pi}\int_0^{1}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \rho^2 \sin\phi \, d\phi\, d\rho\, d\theta$$



4. Chapter 15, problem 12:

4. Chap 15-12

a.



$x = r\cos\theta \quad y = r\sin\theta$

$$\int_0^{a\sin\beta}\int_{y\cot\beta}^{\sqrt{a^2-y^2}} \ln(x^2+y^2)\,dx\,dy$$

$$= \int_0^{\beta}\int_0^{a} \ln r^2 \; r\,dr\,d\theta$$

令 $r^2 = t \quad 2r\,dr = dt$

$$= \int_0^{\beta}\int_0^{a^2} \frac{1}{2}\ln t \; dt\,d\theta$$

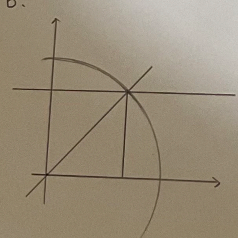
令 $u = \ln t \quad du = \frac{1}{t}dt, \; dv = dt, \; v = t$

$$= \frac{1}{2}\int_0^{\beta}\left(\Big[t\ln t\Big]_0^{a^2} - \int_0^{a^2} t\cdot\frac{1}{t}\,dt\right)d\theta$$

$$= \frac{1}{2}\int_0^{\beta}(2a^2\ln a - a^2)\,d\theta$$

$$= \frac{1}{2}(2a^2\ln a - a^2)\beta = a^2\beta\left(\ln a - \frac{1}{2}\right)$$

b.



对左半: $0 \le x \le a\cos\beta$

$0 \le y \le x\cot\beta$

对右半: $a\cos\beta \le x \le a$

$0 \le y \le \sqrt{a^2-x^2}$

$$\therefore \int_0^{a\sin\beta}\int_{y\cot\beta}^{\sqrt{a^2-y^2}} \ln(x^2+y^2)\,dx\,dy = \int_0^{a\cos\beta}\int_0^{x\cot\beta} \ln(x^2+y^2)\,dy\,dx + \int_{a\cos\beta}^{a}\int_0^{\sqrt{a^2-x^2}} \ln(x^2+y^2)\,dy\,dx$$

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