Brief solutions to selected problems in homework week 02

1. Section 10.3, problem 53:

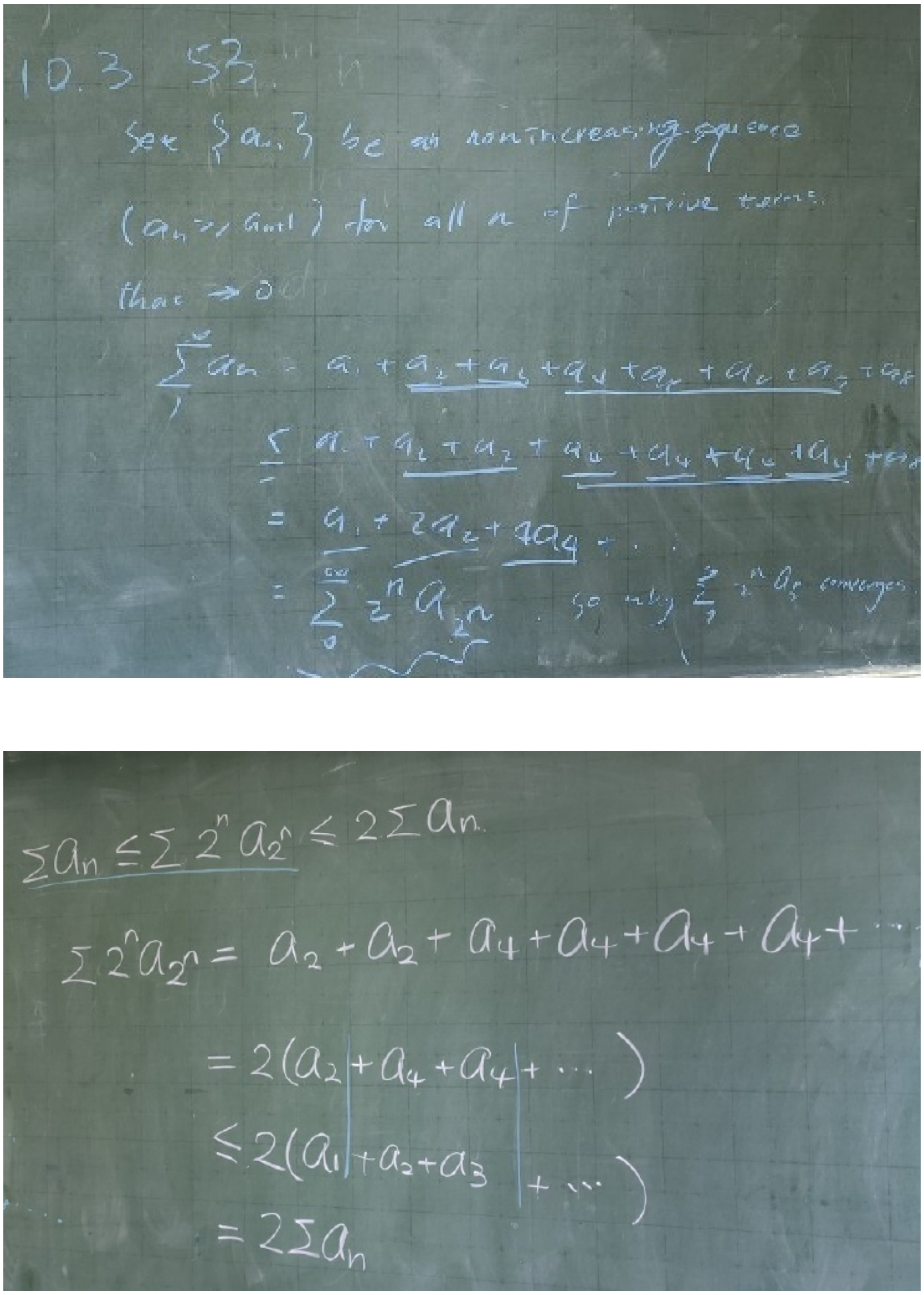

Figure 1: Section 10.3, problem 53

Summary:

Since $a_n \geq a_{n+1} \geq \cdots > 0$, we have

$$a_{2^k} + a_{2^k} + \cdots + a_{2^k} \geq a_{2^k+1} + a_{2^k+2} + \cdots + a_{2^{k+1}} \geq a_{2^{k+1}} + a_{2^{k+1}} + \cdots + a_{2^{k+1}}$$

That is,

$$2^k \cdot a_{2^k} \geq a_{2^k+1} + a_{2^k+2} + \cdots + a_{2^{k+1}} \geq 2^k \cdot a_{2^{k+1}} = \frac{1}{2} 2^{k+1} \cdot a_{2^{k+1}}$$

Summing over k from 0 to ∞, we have

$$\sum_{k=0}^{\infty} 2^k \cdot a_{2^k} \geq \sum_{k=0}^{\infty} (a_{2^k+1} + a_{2^k+2} + \cdots + a_{2^{k+1}}) \geq \frac{1}{2} \sum_{k=0}^{\infty} 2^{k+1} \cdot a_{2^{k+1}} = \frac{1}{2} \sum_{k=1}^{\infty} 2^k \cdot a_{2^k}$$

or

$$\sum_{k=0}^{\infty} 2^k \cdot a_{2^k} \geq \sum_{n=2}^{\infty} a_n \geq \frac{1}{2} \sum_{k=1}^{\infty} 2^k \cdot a_{2^k} \tag{1}$$

From the first inequality in (1), we see that

$$\sum 2^k \cdot a_{2^k} < \infty \implies \sum a_n < \infty$$

and from the second inequality in (1)

$$\sum a_n < \infty \implies \sum 2^k \cdot a_{2^k} < \infty$$

2. Section 10.3, problem 55:

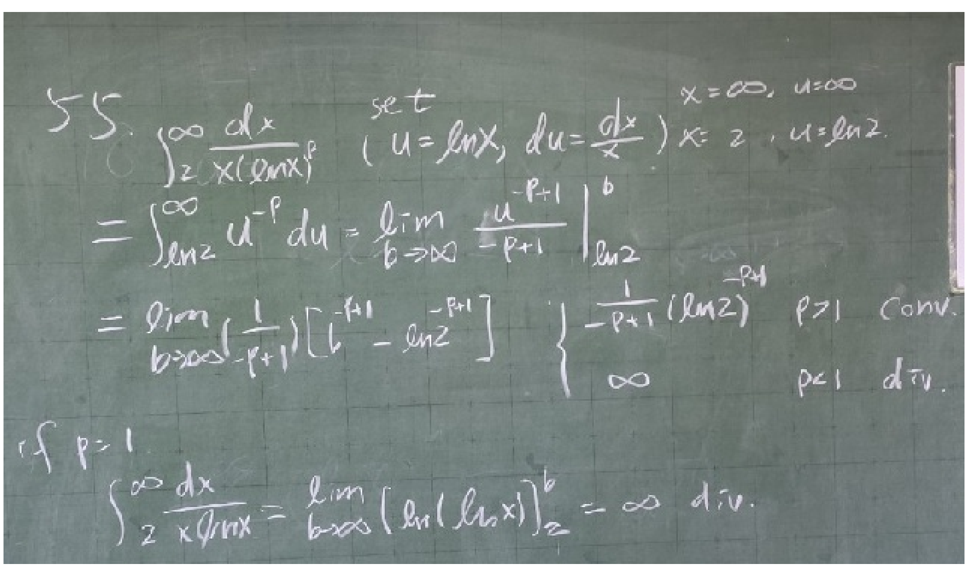

Figure 2: Section 10.3, problem 55