

Brief solutions to selected problems in homework week 02

1. Section 10.3, problem 53:

10.3 53

Let $\{a_n\}$ be an nonincreasing sequence
 $(a_n > a_{n+1})$ for all n of positive terms.
 that $\rightarrow 0$

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + \dots$$

$$\leq a_1 + a_2 + a_2 + a_4 + a_4 + a_4 + a_6 + \dots$$

$$= a_1 + 2a_2 + 2a_4 + \dots$$

$$= \sum_{n=1}^{\infty} 2^n a_{2^n} \quad \text{so why } \sum_{n=1}^{\infty} 2^{-n} a_n \text{ converges}$$

$$\sum a_n \leq \sum 2^n a_{2^n} \leq 2 \sum a_n$$

$$\sum 2^n a_{2^n} = a_2 + a_2 + a_4 + a_4 + a_4 + a_4 + \dots$$

$$= 2(a_2 + a_4 + a_4 + \dots)$$

$$\leq 2(a_1 + a_2 + a_3 + \dots)$$

$$= 2 \sum a_n$$

Figure 1: Section 10.3, problem 53

Summary:

Since $a_n \geq a_{n+1} \geq \dots > 0$, we have

$$a_{2^k} + a_{2^k} + \dots + a_{2^k} \geq a_{2^{k+1}} + a_{2^{k+2}} + \dots + a_{2^{k+1}} \geq a_{2^{k+1}} + a_{2^{k+1}} + \dots + a_{2^{k+1}}$$

That is,

$$2^k \cdot a_{2^k} \geq a_{2^{k+1}} + a_{2^{k+2}} + \dots + a_{2^{k+1}} \geq 2^k \cdot a_{2^{k+1}} = \frac{1}{2} 2^{k+1} \cdot a_{2^{k+1}}$$

Summing over k from 0 to ∞ , we have

$$\sum_{k=0}^{\infty} 2^k \cdot a_{2^k} \geq \sum_{k=0}^{\infty} (a_{2^{k+1}} + a_{2^{k+2}} + \dots + a_{2^{k+1}}) \geq \frac{1}{2} \sum_{k=0}^{\infty} 2^{k+1} \cdot a_{2^{k+1}} = \frac{1}{2} \sum_{k=1}^{\infty} 2^k \cdot a_{2^k}$$

or

$$\sum_{k=0}^{\infty} 2^k \cdot a_{2^k} \geq \sum_{n=2}^{\infty} a_n \geq \frac{1}{2} \sum_{k=1}^{\infty} 2^k \cdot a_{2^k} \tag{1}$$

From the first inequality in (1), we see that

$$\sum 2^k \cdot a_{2^k} < \infty \implies \sum a_n < \infty$$

and from the second inequality in (1)

$$\sum a_n < \infty \implies \sum 2^k \cdot a_{2^k} < \infty$$

2. Section 10.3, problem 55:

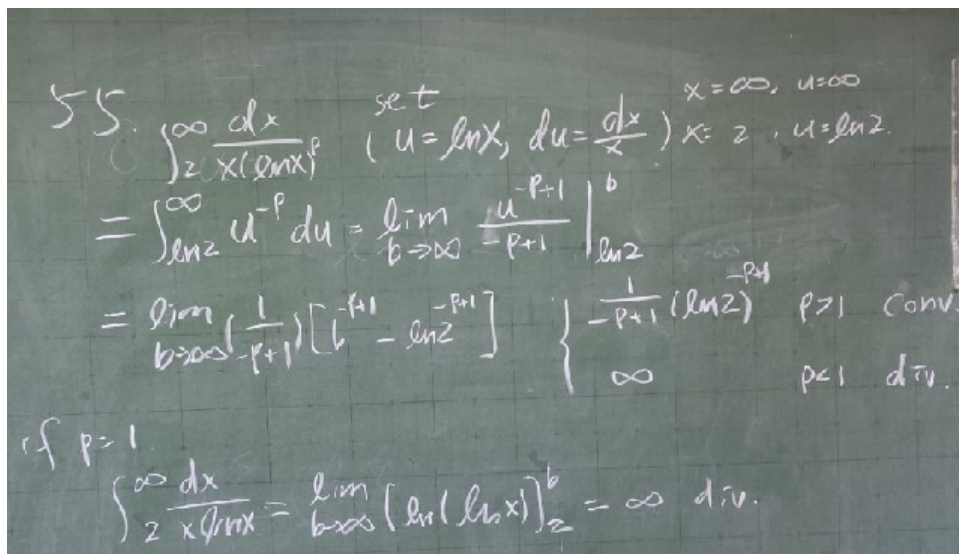


Figure 2: Section 10.3, problem 55