

Brief solutions to selected problems in homework week 02

1. Section 10.3, problem 53:

10.3 53.

Let $\{a_n\}$ be a non-increasing sequence of positive terms such that $a_n \rightarrow 0$.

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= a_1 + \underbrace{a_2 + a_3 + a_4}_{a_6} + \underbrace{a_5 + a_6 + a_7 + a_8}_{a_{10}} + \dots \\ &\leq a_1 + \underbrace{a_2 + a_3}_{a_4} + \underbrace{a_6 + a_7 + a_8 + a_9}_{a_{10}} + \dots \\ &= a_1 + 2a_2 + 4a_4 + \dots \\ &= \sum_{n=0}^{\infty} 2^n a_{2^n}. \end{aligned}$$

so only $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.

$$\begin{aligned} \sum a_n &\leq \sum 2^n a_{2^n} \leq 2 \sum a_n. \\ \sum 2^n a_{2^n} &= a_2 + a_2 + a_4 + a_4 + a_4 + a_4 + \dots \\ &= 2(a_2 + a_4 + a_4 + \dots) \\ &\leq 2(a_1 + a_2 + a_3 + \dots) \\ &= 2 \sum a_n \end{aligned}$$

Figure 1: Section 10.3, problem 53

Summary:

Since $a_n \geq a_{n+1} \geq \dots > 0$, we have

$$a_{2^k} + a_{2^k} + \dots + a_{2^k} \geq a_{2^k+1} + a_{2^k+2} + \dots + a_{2^{k+1}} \geq a_{2^{k+1}} + a_{2^{k+1}} + \dots + a_{2^{k+1}}$$

That is,

$$2^k \cdot a_{2^k} \geq a_{2^k+1} + a_{2^k+2} + \dots + a_{2^{k+1}} \geq 2^k \cdot a_{2^{k+1}} = \frac{1}{2} 2^{k+1} \cdot a_{2^{k+1}}$$

Summing over k from 0 to ∞ , we have

$$\sum_{k=0}^{\infty} 2^k \cdot a_{2^k} \geq \sum_{k=0}^{\infty} (a_{2^k+1} + a_{2^k+2} + \dots + a_{2^{k+1}}) \geq \frac{1}{2} \sum_{k=0}^{\infty} 2^{k+1} \cdot a_{2^{k+1}} = \frac{1}{2} \sum_{k=1}^{\infty} 2^k \cdot a_{2^k}$$

or

$$\sum_{k=0}^{\infty} 2^k \cdot a_{2^k} \geq \sum_{n=2}^{\infty} a_n \geq \frac{1}{2} \sum_{k=1}^{\infty} 2^k \cdot a_{2^k} \quad (1)$$

From the first inequality in (1), we see that

$$\sum 2^k \cdot a_{2^k} < \infty \implies \sum a_n < \infty$$

and from the second inequality in (1)

$$\sum a_n < \infty \implies \sum 2^k \cdot a_{2^k} < \infty$$

2. Section 10.3, problem 55:

The handwritten solution shows the evaluation of the improper integral $\int_2^\infty \frac{dx}{x(\ln x)^p}$. It uses the substitution $u = \ln x$, $du = \frac{dx}{x}$. The limits of integration change to $x=2$ and $u=\ln 2$. The integral becomes $\int_{\ln 2}^\infty u^{-p} du$. This is evaluated as $\lim_{b \rightarrow \infty} \left[\frac{u^{-p+1}}{-p+1} \right]_{\ln 2}^b$. The result is $\lim_{b \rightarrow \infty} \left(\frac{1}{-p+1} \left[\frac{1}{b} - \ln 2 \right] \right)$. For $p > 1$, the term $\frac{1}{-p+1} \ln 2$ is finite, so the limit goes to zero, indicating convergence. For $p \leq 1$, the term $\frac{1}{-p+1}$ is either infinite or zero, leading to divergence. Below this, a separate calculation shows that if $p=1$, the integral $\int_2^\infty \frac{dx}{x \ln x}$ diverges, as the limit of $\ln(\ln x)$ as $x \rightarrow \infty$ is infinity.

Figure 2: Section 10.3, problem 55