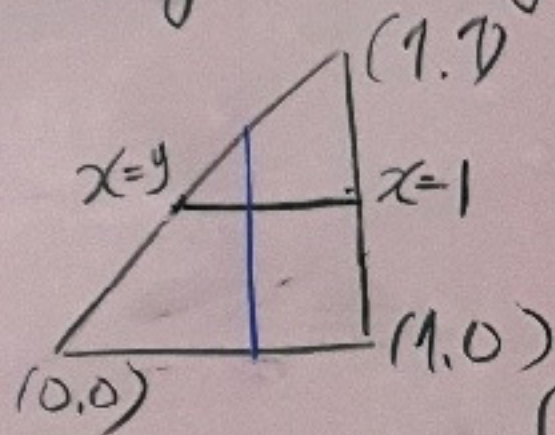


Example : R = region bounded
by $x=y$, $x=1$, $y=0$



$$\iiint_R \frac{\sin x}{x} dA = ?$$

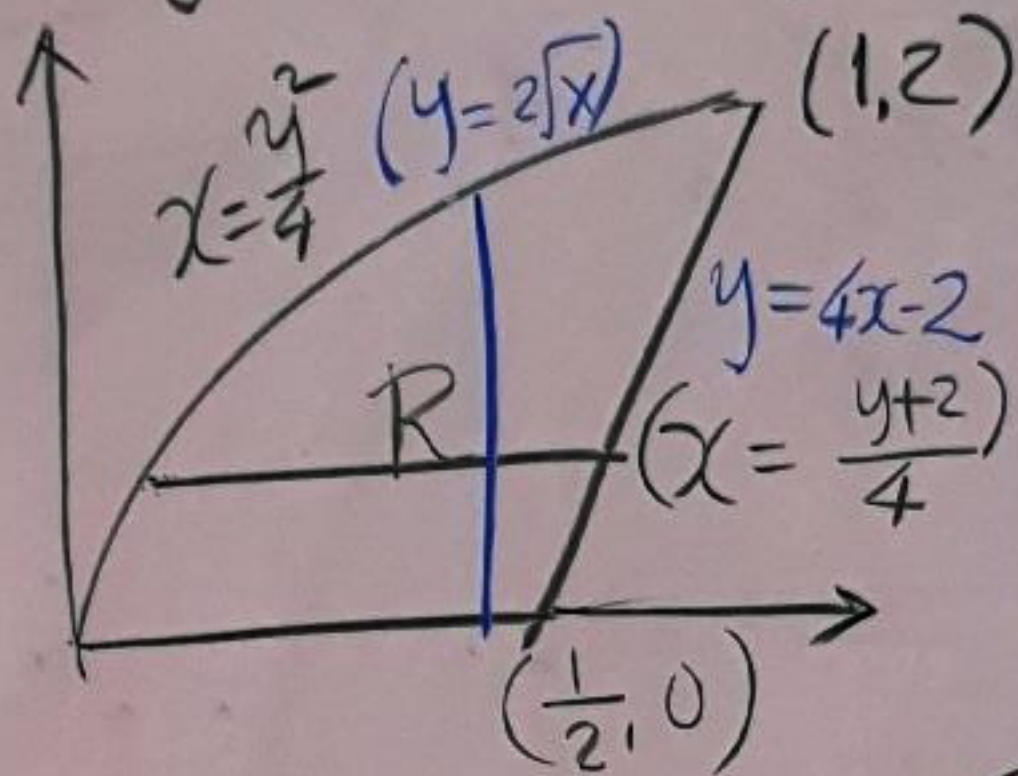
(i) $\iint dx dy$: $\int_0^1 \int_y^1 \frac{\sin x}{x} dx dy = ?$

(ii) $\iint dy dx$: $\int_0^1 \int_0^x \frac{\sin x}{x} dy dx$

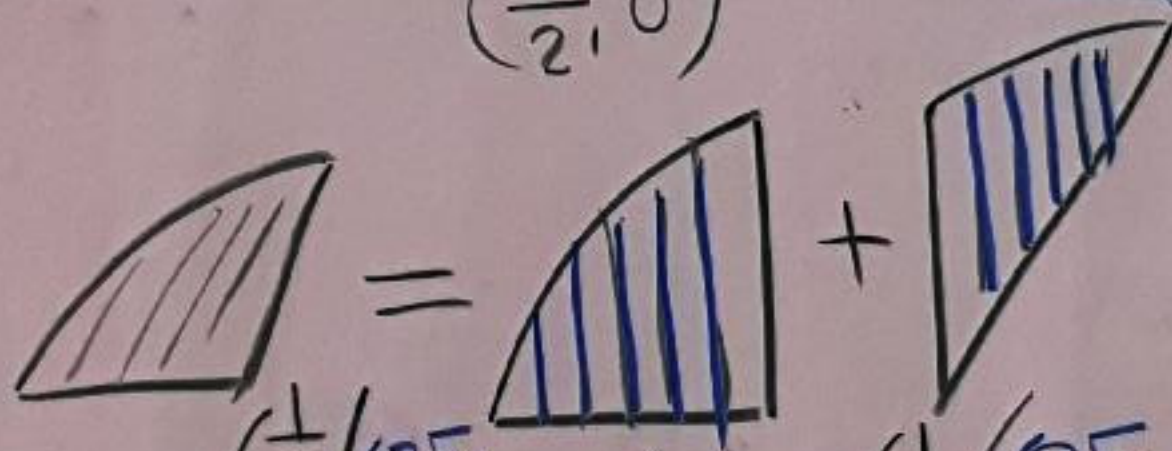
$$= \int_0^1 (x-0) \frac{\sin x}{x} dx = 1 - \cos x$$

Example Find area bounded

by $y=0$, $y=4x-2$, $x=\frac{y^2}{4}$



$$\text{Area of } R = \iint_R 1 \, dA$$

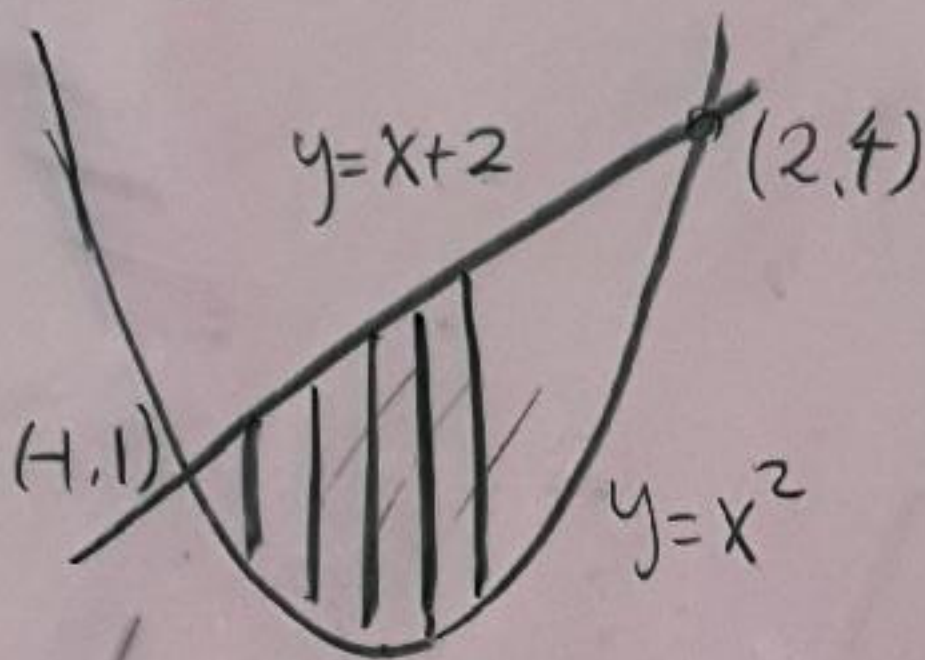


$$= \int_0^{\frac{1}{2}} \left(\int_0^{2\sqrt{x}} 1 \, dy \right) dx + \int_{\frac{1}{2}}^1 \left(\int_{4x-2}^{2\sqrt{x}} 1 \, dy \right) dx$$

or

$$= \int_0^2 \left(\int_{\frac{y^2}{4}}^{\frac{y+2}{4}} 1 \, dx \right) dy$$

Example Find area bounded
by $y = x^2$ and $y = x + 2$



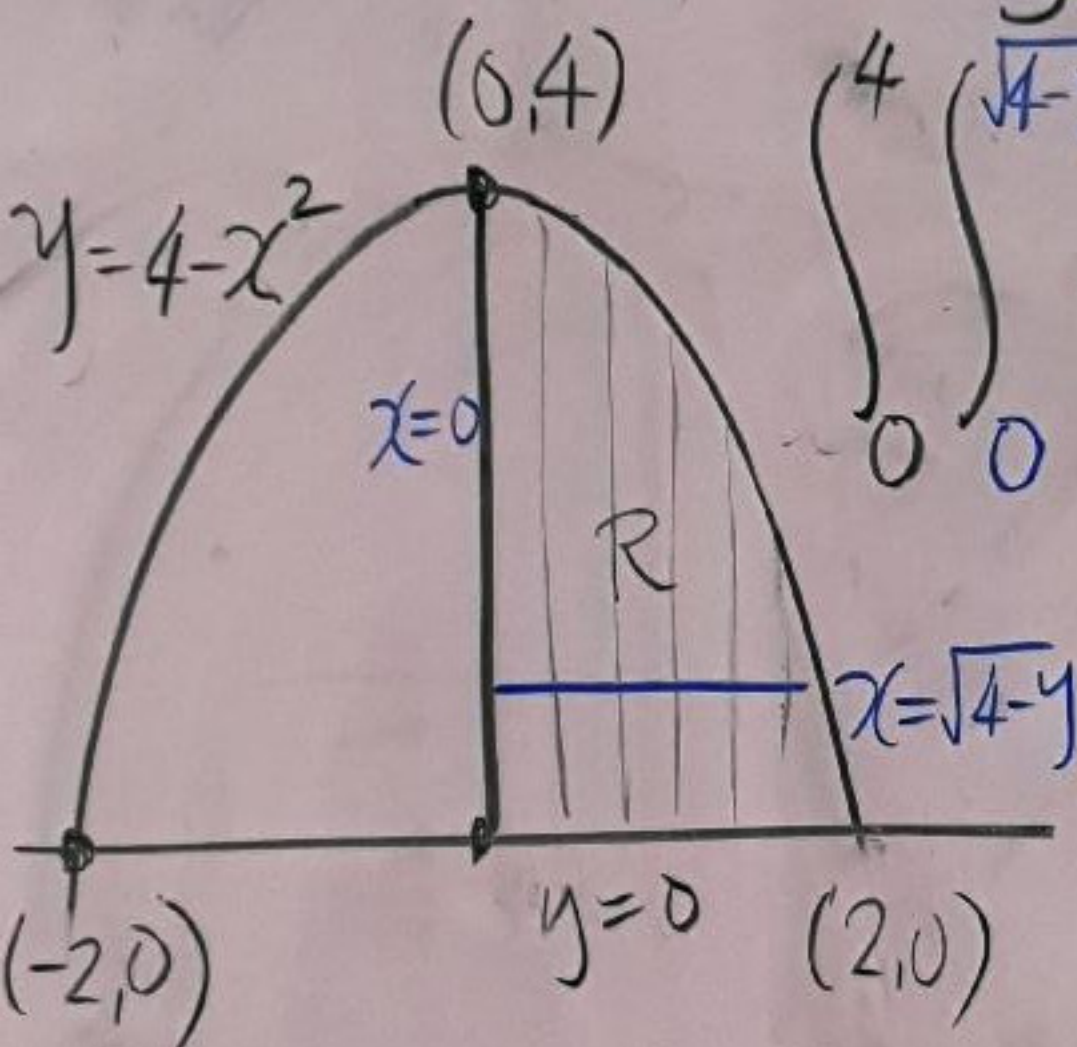
$$\int_{-1}^2 \int_{x^2}^{x+2} 1 \, dy \, dx$$

$$= \int_{-1}^2 (x+2-x^2) \, dx = \frac{9}{2}$$

Example $\int_0^2 \int_0^{4-x^2} \frac{x e^{2y}}{4-y} dy dx = ?$

difficult

Sol: Try $dx dy$:



$$\int_0^4 \int_0^{\sqrt{4-y}} \frac{x e^{2y}}{4-y} dx dy$$

$$= \int_0^4 \frac{e^{2y}}{4-y} \cdot \left(\frac{x^2}{2} \right) \Big|_0^{\sqrt{4-y}} dy$$

$$= \frac{1}{2} \int_0^4 e^{2y} dy$$

$$= \frac{1}{4} e^{2y} \Big|_0^4 = \frac{e^8 - 1}{4}$$

Integration in polar coordinate

Eg: $\iint_R f(x,y) dA$, $R = \text{region bounded by } y=x, x^2+y^2=1, y=\sqrt{3}x, x^2+y^2=4$



Both $\int_x^* \int_y^* f(x,y) dx dy$

and $\int_x^* \int_y^* f(x,y) dy dx$

are complicated (need to break into several integrals)

In this example

$$R = \left\{ (r, \theta) \mid 1 \leq r \leq 2, \frac{\pi}{4} \leq \theta \leq \frac{\pi}{3} \right\}$$

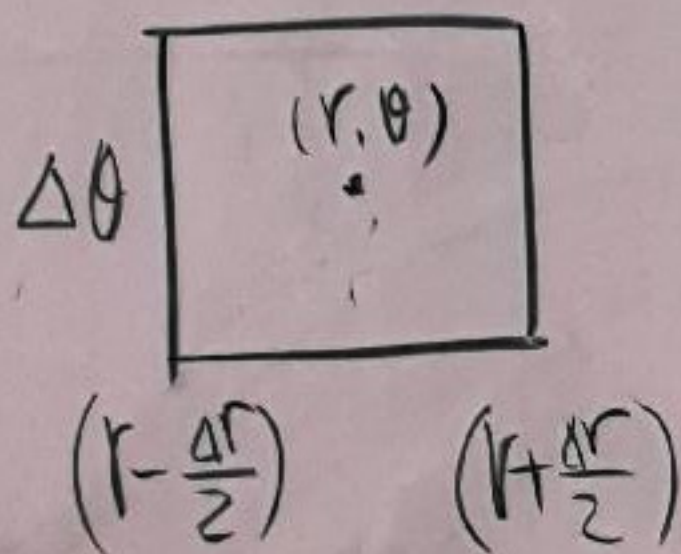
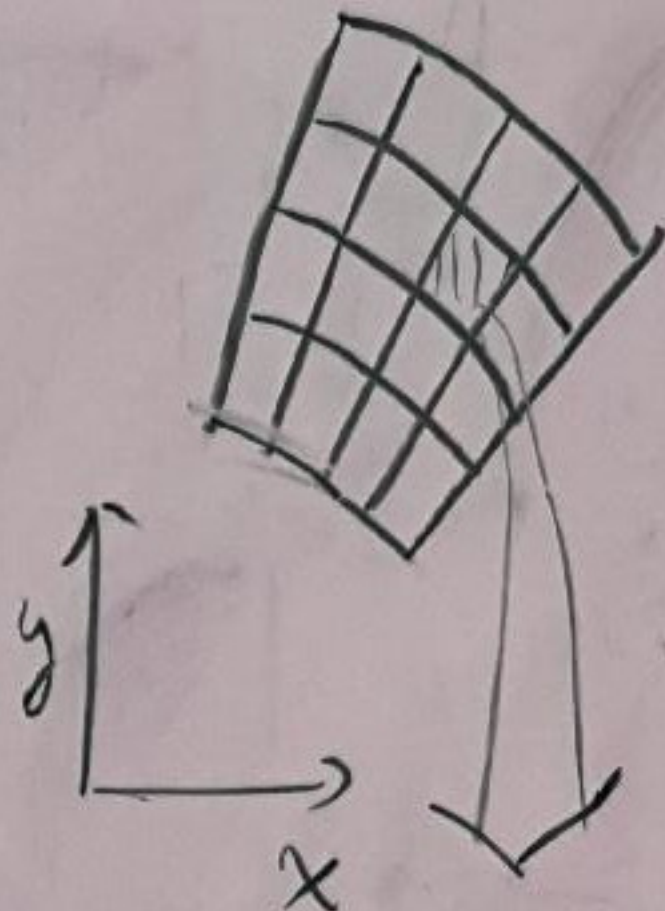
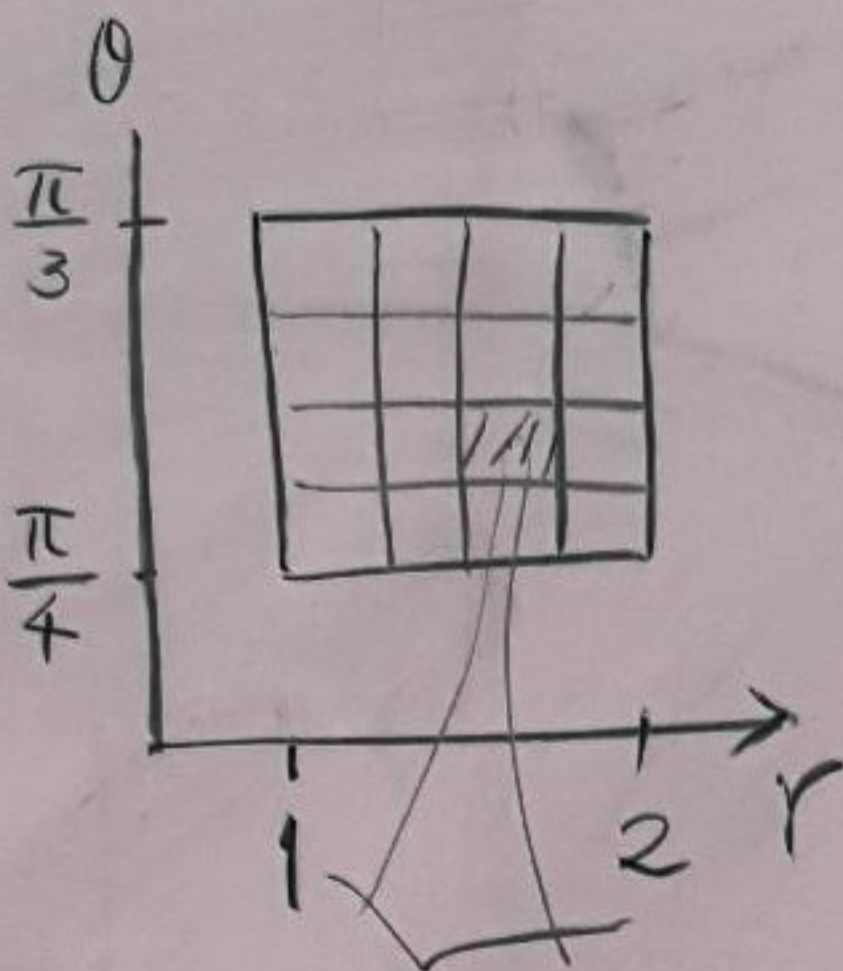
So either $dr d\theta$ or $d\theta dr$
are convenient

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \int_{r=1}^2 f(\overset{x}{r \cos \theta}, \overset{y}{r \sin \theta}) dA$$

$dA = ? dr d\theta$

Ans: $dA = r dr d\theta$

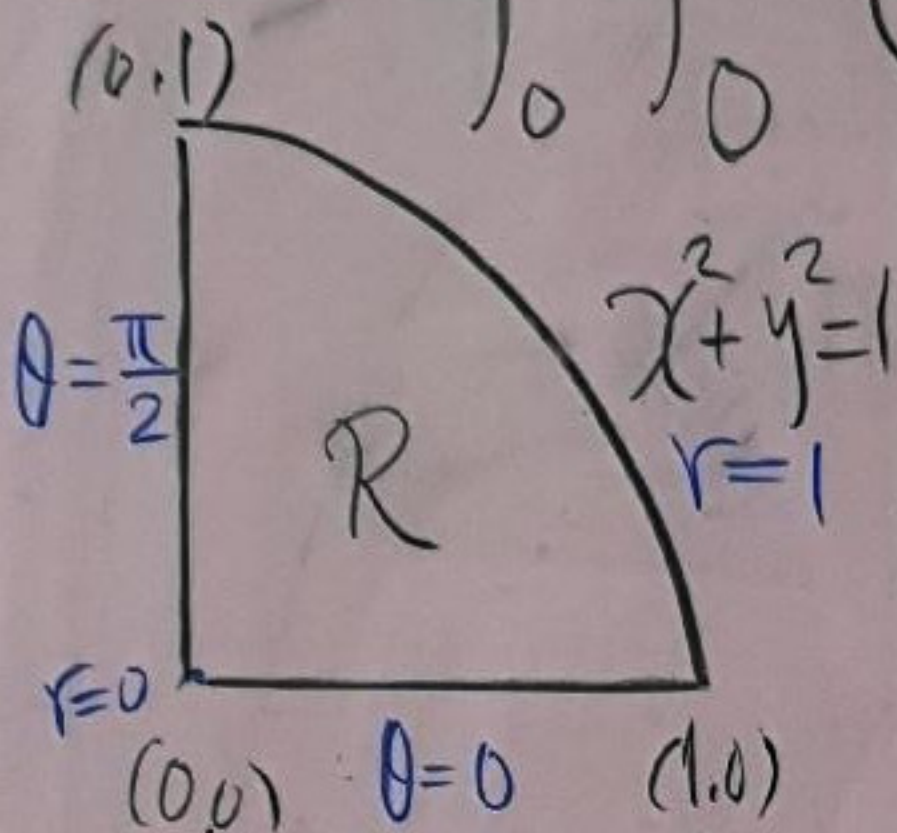
$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \int_1^2 f(r \cos \theta, r \sin \theta) r dr d\theta$$



$$\Delta A = \frac{(r + \frac{dr}{2})^2 \Delta \theta}{2} - \frac{(r - \frac{dr}{2})^2 \Delta \theta}{2}$$

$$= r \Delta r \Delta \theta \Rightarrow dA = r dr d\theta$$

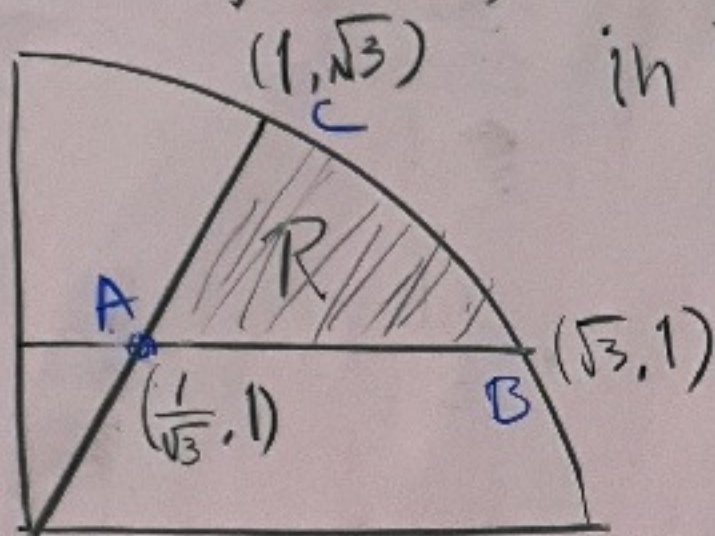
Eg: $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx$



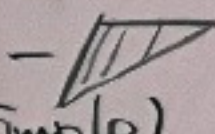
$$= \int_0^{\frac{\pi}{2}} \int_0^1 r^2 \cdot \underbrace{r dr d\theta}$$

$$= \frac{\pi}{2} \cdot \frac{1}{4} = \frac{\pi}{8}$$

Example: Find area bounded
by $y = \sqrt{3}x$, $x^2 + y^2 = 4$, $y = 1$ in
first quadrant



Method 1:

Area =  -  (Simple)

Method 2: Area = $\int_1^{\sqrt{3}} \int_{\frac{y}{\sqrt{3}}}^{\sqrt{4-y^2}} 1 \, dx \, dy =$
= Complicated

From Method 1: $A = (\pi - \sqrt{3})/3$

Method 3: (polar coord)

$$y = \sqrt{3}x, \Leftrightarrow \theta = \frac{\pi}{3}$$

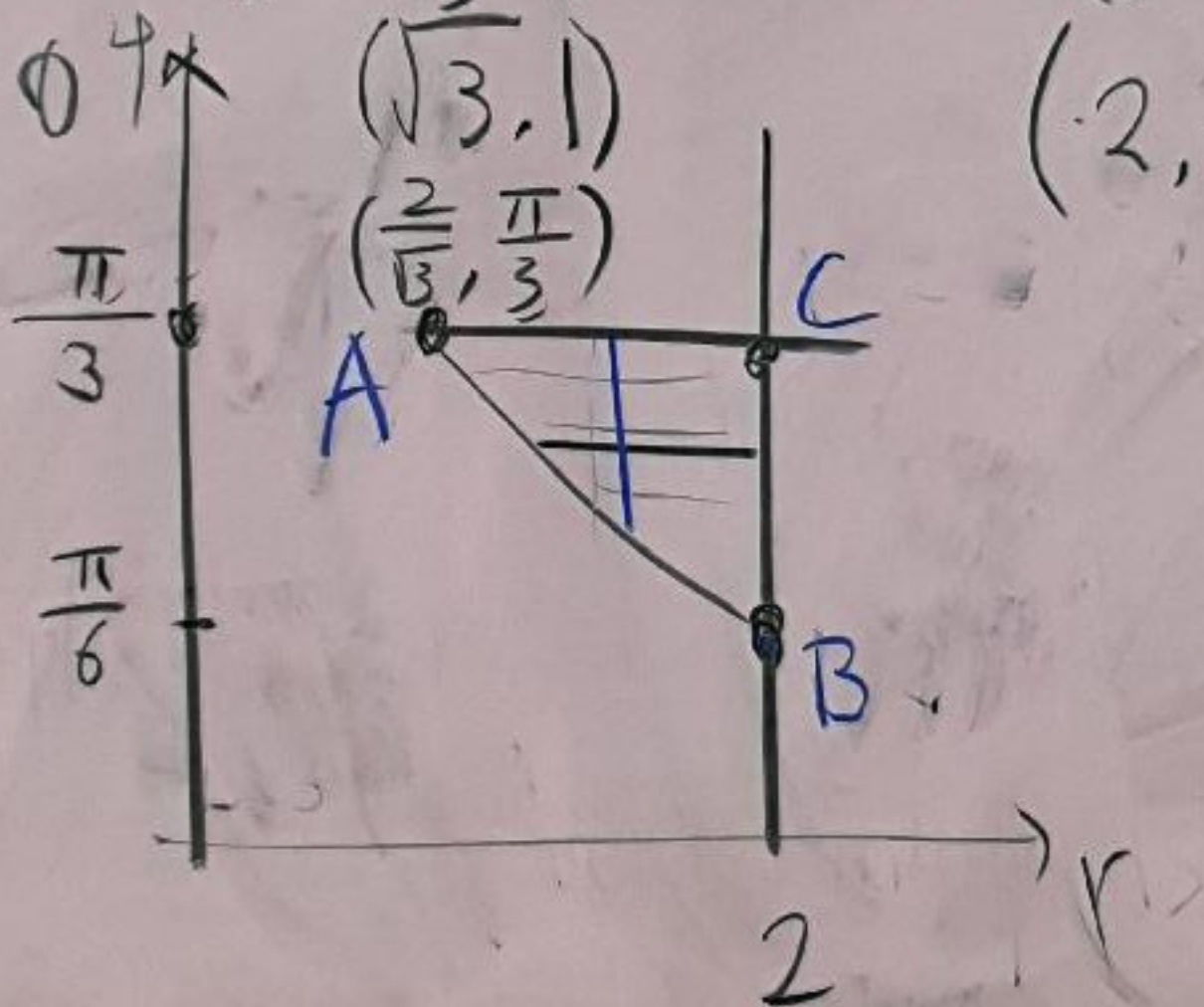
$$x^2 + y^2 = 4 \Leftrightarrow r = 2$$

$$r = \csc \theta$$

$$y = 1 \Leftrightarrow r \sin \theta = 1 \text{ or } \theta = \sin^{-1}\left(\frac{1}{r}\right)$$

$$(x, y) = \left(\frac{1}{\sqrt{3}}, 1\right) \Leftrightarrow (r, \theta) = \left(\frac{2}{\sqrt{3}}, \frac{\pi}{3}\right)$$

$$\left(2, \frac{\pi}{6}\right)$$



$$\text{Area} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_{\csc \theta}^2 r \, dr \, d\theta \dots (I)$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{4 - \csc^2 \theta}{2} \right) d\theta$$

(see textbook for

(= ~~more~~ complicated) details of
not very

integration)

$$A = \int_{\frac{2}{\sqrt{3}}}^2 r \int_{\sin^{-1}(\frac{1}{r})}^{\frac{\pi}{3}} 1 \, d\theta \, dr \dots (II)$$

$$= \int_{\frac{2}{\sqrt{3}}}^2 r \left(\frac{\pi}{3} - \sin^{-1}\left(\frac{1}{r}\right) \right) dr$$

(most complicated)