Divergence Theorem

S: piecewise smooth oriented closed surface

D: inside of S ($S = \partial D$)

$\vec{F}: D \longrightarrow \mathbb{R}^3$ is continously differentiable (F_i, $\partial_j F_i$ are all cont.)

Then

$$\iiint_D \nabla \cdot \vec{F} \, dV = \iint_S \vec{F} \cdot \vec{n} \, d\sigma$$

where $\nabla \cdot \vec{F} = \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3$

$\underline{\underline{\text{Pf}}}$: $\vec{F} = (M, 0, 0)$.

Case I: $D = [\underline{x}, \overline{x}] \times [\underline{y}, \overline{y}] \times [\underline{z}, \overline{z}]$

$\operatorname{div} \vec{F} = \partial_x M$.

$$\iiint_D = \int_{\underline{z}}^{\overline{z}} \int_{\underline{y}}^{\overline{y}} \int_{\underline{x}}^{\overline{x}} M_x(x, y, z)\, dx\, dy\, dz$$

$$= \iint \left(M(\overline{x}, y, z) - M(\underline{x}, y, z) \right) dy\, dz$$

∂D consist of six parts:

$x = \underline{x}$, $x = \overline{x}$, $y = \underline{y}$, $y = \overline{y}$, $z = \underline{z}$, $z = \overline{z}$

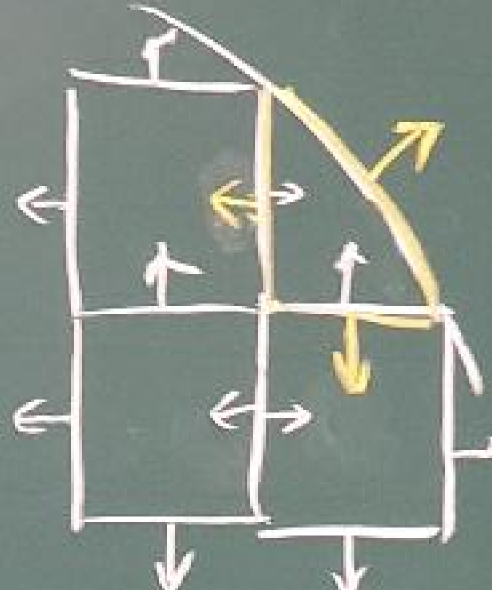
$\vec{n} = (-1, 0, 0), (1, 0, 0)(0, -1, 0), (0, 1, 0), (0, 0, -1), (0, 0, 1)$

$\vec{F} \cdot \vec{n}$: $-M$, M, 0, 0, 0, 0

$$= \int_{\underline{z}}^{\overline{z}} \int_{\underline{y}}^{\overline{y}} \vec{F} \cdot \vec{n}(\overline{x}, y, z)\, dy\, dz + \int_{\underline{z}}^{\overline{z}} \int_{\underline{y}}^{\overline{y}} \vec{F} \cdot \vec{n}(\underline{x}, y, z)\, dy\, dz$$

$$= \iint_{\partial D} \vec{F} \cdot \vec{n}\, d\sigma$$

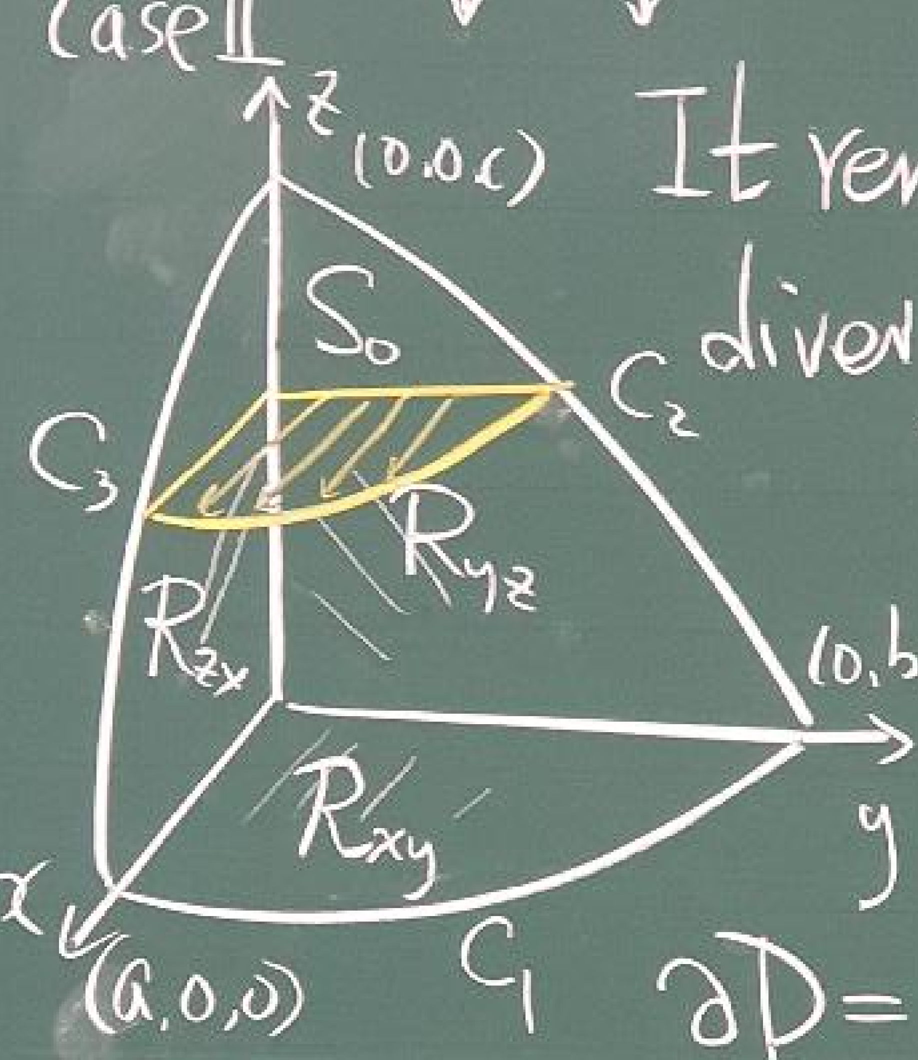
Recall 2D case



Superposition principle (on domains) for G_T, G_N

$\Rightarrow G_T, G_N$ on general domains

Case II



It remains to verify divergence Thm on domains like this one.

$$\partial D = S_0 \cup R_{xy} \cup R_{yz} \cup R_{zx}$$

where
$$\begin{aligned} S_0 &= \{ z = f(x,y),\ x,y \in R_{xy} \} \\ &= \{ x = g(y,z),\ y,z \in R_{yz} \} \\ &= \{ y = h(z,x),\ z,x \in R_{zx} \} \end{aligned}$$

where
$$\begin{aligned} R_{xy} &= \{ 0 \le y \le h(0,x),\ 0 \le x \le a \} \\ &= \{ 0 \le x \le g(y,0),\ 0 \le y \le b \} \end{aligned}$$

and $C_1 = \{ y = h(0,x),\ 0 \le x \le a \} = \{ x = g(y,0),\ 0 \le y \le b \}$

Similarly for R_{yz}, R_{zx}, C_2, C_3.

$$\iiint_D M_x\,dV = \int_{z=0}^{c}\int_{y=0}^{h(z,0)}\int_{x=0}^{g(y,z)} M_x\,dx\,dy\,dz$$

Note: $R_{yz} = \{0 \le z \le f(0,y),\ 0 \le y \le b\}$
$= \{0 \le y \le h(z,0),\ 0 \le z \le c\}$

$C_2 = \{z = f(0,y),\ 0 \le y \le b\} = \{y = h(z,0),\ 0 \le z \le c\}$

$$= \underbrace{\iint_{R_{yz}} \underbrace{M(g(y,z),y,z)\,dy\,dz}_{\vec{F}\cdot\vec{n}\,d\sigma \text{ on } S_0} + \iint_{R_{yz}} \underbrace{-M(0,y,z)\,dy\,dz}_{\vec{F}\cdot\vec{n}\,d\sigma \text{ on } R_{yz}}}_{= \iint_{\partial D} \vec{F}\cdot\vec{n}\,d\sigma}$$

	S_0	R_{xy}	R_{yz}	R_{zx}
$\vec{n}$:	$\vec{n}_0$	$(0,0,-1)$	$(-1,0,0)$	$(0,-1,0)$
$\vec{n}\,d\sigma$:	$\begin{pmatrix}1\\-g_y\\-g_z\end{pmatrix} dy\,dz$	$\begin{pmatrix}0\\0\\-1\end{pmatrix} dx\,dy$	$\begin{pmatrix}-1\\0\\0\end{pmatrix} dy\,dz$	$\begin{pmatrix}0\\-1\\0\end{pmatrix} dx\,dz$

$$\vec{n}_0 = \pm\frac{\nabla(x - g(y,z))}{|\ \text{"}\ |} \quad \frac{\vec{n}_0\cdot\hat{x} > 0}{\text{"}+\text{"}} \quad \frac{(1, -g_y, -g_z)}{\sqrt{1+g_y^2+g_z^2}}$$

$$d\sigma = \frac{|\vec{n}_0|}{|\vec{n}_0\cdot\hat{x}|}\,dy\,dz = \sqrt{1+g_y^2+g_z^2}\,dy\,dz$$

Note: $\vec{F}\cdot\vec{n} = 0$ on R_{xy} and R_{zx}

Eg $D = \{x^2+y^2+z^2 \le 4,\ z \ge 0\}$

$\partial D = S_0 \cup R$

$S_0 = \{x^2+y^2+z^2 = 4,\ z \ge 0\},\ \vec{n} = \frac{(x,y,z)}{2}$

$R = \{x^2+y^2 \le 4,\ z = 0\}.\ \vec{n} = (0,0,-1)$

Eg: $\vec{F} = \nabla\left(-\frac{1}{\rho}\right) = -\nabla(x^2+y^2+z^2)^{-\frac{1}{2}} = \frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}}$

$\nabla \cdot \vec{F} = 0$ (check!), $S = \{x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1\}$

(2D analogue: $\vec{F} = \nabla \ln r = \nabla \ln(x^2+y^2)^{\frac{1}{2}}$)

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = ?$$

Sol: Method 1: $S = \underset{z>0}{S^+} \cup \underset{z<0}{S^-}$. $S^+ = \left\{ z = \sqrt{9 - 9x^2 - \frac{9y^2}{4}},\ x^2 + \frac{y^2}{4} \le 1 \right\}$

$\cancel{\vec{n}\, d\sigma = \left(2x,\ \frac{y}{2},\ \frac{2}{9}\sqrt{9-9x^2-\frac{9y^2}{4}}\right) dA}$

$\cancel{\vec{F}\cdot\vec{n}\, d\sigma = \left(2x^2 + \frac{y^2}{2} + \frac{2}{9}\left(9-9x^2-\frac{9}{4}y^2\right)\right) \Big/ \left(x^2+y^2+9-9x^2-\frac{9}{4}y^2\right)^{\frac{3}{2}} dA}$

Method I: $\underset{(z>0)}{S = S^{+}} \cup \underset{(z<0)}{S^{-}}$

$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = 2\iint_{S^+} \vec{F}\cdot\vec{n}\, d\sigma$$

$$\left(\begin{array}{l}\text{due to symmetry:}\\ \vec{F}\cdot\vec{n}\big|_{(x,y,-z)} = \vec{F}\cdot\vec{n}\big|_{(x,y,z)}\\ d\sigma\big|_{(x,y,-z)} = d\sigma\big|_{(x,y,z)}\end{array}\right)$$

On S^+: (let $g(x,y,z) = x^2 + \frac{y^2}{4} + \frac{z^2}{9}$)

$$\vec{n} = \frac{+\nabla g}{|\nabla g|},\quad d\sigma = \frac{|\nabla g|}{|\nabla g\cdot\hat{z}|}\, dA$$

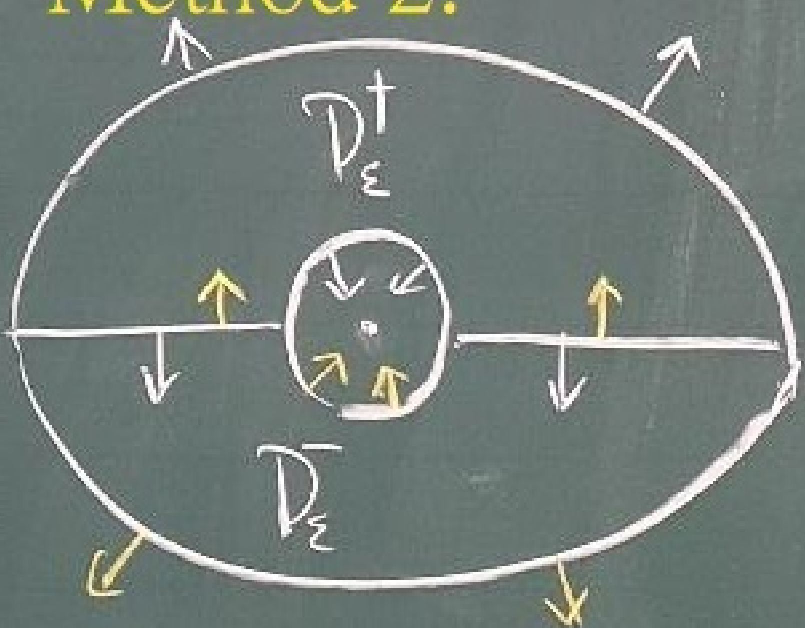
$$R = \left\{x^2 + \frac{y^2}{4} < 1\right\},\quad z = 3\left(1 - x^2 - \frac{y^2}{4}\right)^{\frac{1}{2}}$$

$$\Rightarrow \iint_S = 2\iint_{S^+} = 2\iint_R \frac{2x^2 + \frac{y^2}{2} + \frac{2z^2}{9}}{(x^2+y^2+z^2)^{3/2}\,\frac{2z}{9}}\, dA$$

$$= 2\iint_{x^2+\frac{y^2}{4}<1} \frac{2x^2 + \frac{y^2}{2} + 2\left(1 - x^2 - \frac{y^2}{4}\right)}{x^2+y^2+9\left(1-x^2-\frac{y^2}{4}\right)\frac{2\cdot 3}{9}\sqrt{1-x^2-\frac{y^2}{4}}}\, dx\,dy$$

$$\cancel{\iint_S \vec{F}\cdot\vec{n}\,d\sigma = 2\iint_{x^2+\frac{y^2}{4}\le 1} \frac{2+x^2}{\left(9-8x^2-\frac{5y^2}{4}\right)^{\frac{3}{2}}}\,dA}$$

Method 2:



$$D^{\pm}_\varepsilon = \left\{ \begin{array}{c} x^2+\frac{y^2}{4}+\frac{z^2}{9}\le 1 \\ x^2+y^2+z^2 \ge \varepsilon^2 \\ z \gtrless 0 \end{array} \right\}$$

$$D_\varepsilon = D^+_\varepsilon \cup D^-_\varepsilon$$

$$\partial D_\varepsilon = S \cup \underset{\vec{n}=\frac{-(x,y,z)}{\varepsilon}}{S_\varepsilon} \qquad S_\varepsilon = \{x^2+y^2+z^2=\varepsilon^2\}$$

Div. Thm:

$$0 = \iiint_{D_\varepsilon} \nabla\cdot\vec{F}\,dV = \iint_S \vec{F}\cdot\vec{n}\,d\sigma - \iint_{S_\varepsilon} \vec{F}\cdot\underset{\frac{(x,y,z)}{\varepsilon}}{\underset{\parallel}{\vec{n}}}\,d\sigma$$

$$\therefore \iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iint_{S_\varepsilon} \frac{\vec{r}}{\varepsilon^3}\cdot\frac{\vec{r}}{\varepsilon}\,d\sigma = \int_0^{2\pi}\int_{\phi=0}^{\pi} \frac{1}{\varepsilon^2}\underbrace{\varepsilon^2\sin\phi\,d\phi\,d\theta}_{\text{Area of } S_\varepsilon}$$

$$= \frac{1}{\varepsilon^2}4\pi\varepsilon^2 = 4\pi$$

Rm S: smooth oriented closed surface

$$\vec{F} = -\nabla\left(\frac{-1}{\rho}\right)$$

$$\Rightarrow \iint_S \vec{F}\cdot\vec{n}\, d\sigma = \begin{cases} 0 & , \vec{0} \in \text{outside of } S \\ 4\pi & , \vec{0} \in \text{inside.} \end{cases}$$

also $\displaystyle\iint_{S_\varepsilon} \vec{F}\cdot\vec{n}\, d\sigma = 4\pi$ indep. of $\varepsilon > 0$

Rm If D has holes,

$$\Rightarrow \vec{n} = \begin{cases} \text{outward, at outer boundary} \\ \text{inward of holes at inner bdry} \end{cases}$$

Rm Recall: $\nabla\times\nabla f = \vec{0}$

Here, $$\nabla\cdot\nabla\times\vec{F} = \begin{pmatrix}\partial_x\\ \partial_y\\ \partial_z\end{pmatrix}\cdot\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial_x & \partial_y & \partial_z\\ F_1 & F_2 & F_3\end{vmatrix}$$

$$= \begin{vmatrix}\partial_x & \partial_y & \partial_z\\ \partial_x & \partial_y & \partial_z\\ F_1 & F_2 & F_3\end{vmatrix} = 0$$