Stokes' Thm.

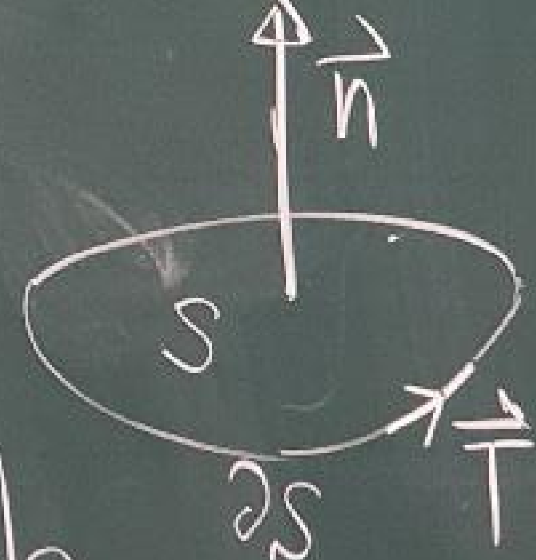
S: a smooth oriented surface

$\vec{F}$ and 1st derivatives of $\vec{F}$ are continuous in an open region containing S.

Then

$$\oint_{\partial S} \vec{F}\cdot\vec{T}\,ds = \iint_{S} \nabla\times\vec{F}\cdot\vec{n}\,d\sigma$$

where $\vec{n}$ and $\vec{T}$ follow the right hand rule



Eg 4.5: $\vec{F} = (x^2 - y, 4z, x^3)$

$S = \left\{ z = \sqrt{x^2+y^2},\ \begin{matrix} 0 \le z \le 2 \\ (\text{i.e. } x^2+y^2 \le 4) \end{matrix} \right\}$

$\iint_S \nabla \times \vec{F} \cdot \vec{n}\, d\sigma = ?$ $\vec{n}$ = upward

"$C = \cos\theta,\ S = \sin\theta$"

Sol $\vec{r}(r,\theta) = (rC, rS, r)$

$\vec{n}\, d\sigma = \pm\, \vec{r}_r \times \vec{r}_\theta\, dr\, d\theta$

$= \pm \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ C & S & 1 \\ -rS & rC & 0 \end{vmatrix} dr\, d\theta \underset{\text{"+"}}{=} \begin{pmatrix} -rC \\ -rS \\ r \end{pmatrix} dr\, d\theta$

$(d\sigma = \sqrt{2}\, r\, dr\, d\theta = \sqrt{2}\, dA)$

$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 - y, & 4z, & x^2 \end{vmatrix} = \begin{pmatrix} -4 \\ -2x \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -2rC \\ 1 \end{pmatrix}$

Method I:

$$\iint_S \nabla\times\vec{F}\cdot\vec{n}\,d\sigma = \iint_{x^2+y^2\le 4} (4rC + 2r^2CS + r)\,dr\,d\theta$$

$$= \int_{r=0}^{2}\int_{\theta=0}^{2\pi} (4rC + 2r^2CS + r)\,d\theta\,dr$$

$$= \int_0^2 r\,dr \int_0^{2\pi} d\theta = 4\pi$$

Method II: $\partial S = \{x^2+y^2=4,\ z=2\}$

$$\vec{r}(\theta) = (2C, 2S, 2)$$

$$\vec{T}\,ds = \frac{d\vec{r}}{d\theta}\,d\theta = (-2S, 2C, 0)\,d\theta$$

$$\overset{\text{Stokes}}{=} \int_{\partial S} \vec{F}\cdot\vec{T}\,ds = \int_{\theta=0}^{2\pi} \begin{pmatrix} 4C^2-2S \\ 8 \\ 4C^2 \end{pmatrix}\cdot\begin{pmatrix} -2S \\ 2C \\ 0 \end{pmatrix} d\theta$$

$$= \int_0^{2\pi} 4S^2\,d\theta = 2\pi\cdot 2 = 4\pi$$

Method III

$$\iint_S \nabla\times\vec{F}\cdot\vec{n}\,d\sigma = \oint_{\partial S}\vec{F}\cdot\vec{T}\,ds = \iint_{\tilde{S}} \nabla\times\vec{F}\cdot\vec{n}\,d\sigma$$

for any $\tilde{S}$ with $\partial\tilde{S} = \partial S$

Take $\tilde{S} = \{x^2+y^2 \leq 4,\ z=2\}$

$$\vec{r}(r,\theta) = (rC, rS, 2)$$

$$d\sigma = dA = r\,dr\,d\theta$$

$$\nabla\times\vec{F}\cdot\vec{n} = \begin{pmatrix}-4\\-2rC\\1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = 1$$

$$\iint_{\tilde{S}} = \int_0^{2\pi}\int_0^2 1\,r\,dr\,d\theta = 4\pi$$

Eg6: $S = \{ z = y^2 - x^2 ,\ x^2 + y^2 \le 1 \}$

$\vec{F} = (y , -x , x^2)$

Verify the Stokes' Thm on S.

<u>Sol</u> $\vec{n}$ = upward $\Leftrightarrow$ $\vec{T} = \circlearrowleft$

(Top view)

$$\partial S : \vec{r}(\theta) = (C , S , S^2 - C^2)$$

$$\vec{T}\, ds = \frac{d\vec{r}}{d\theta}\, d\theta = (-S , C , 4SC)$$

$$\oint_{\partial S} = \int_0^{2\pi} \begin{pmatrix} S \\ -C \\ C^2 \end{pmatrix} \cdot \begin{pmatrix} -S \\ C \\ 4SC \end{pmatrix} d\theta$$

$$= \int_0^{2\pi} (-1 + 4SC^3)\, d\theta = -2\pi + \int_0^{2\pi} 4SC^3\, d\theta$$

$$\int_0^{2\pi} 4SC^3\, d\theta = -\cancel{4}\int_0^{2\pi} C^3\, dC = -\int_0^{2\pi} dC^4 = 0$$

$$S:\ \vec{r}(r,\theta)=(rc,\ rs,\ r^2(s^2-c^2))$$

$$0\le r\le 1,\quad 0\le\theta\le 2\pi$$

$$\vec{n}\,d\sigma=\pm\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ c & s & 2r(s^2-c^2)\\ -rs & rc & 4r^2sc\end{vmatrix}dr\,d\theta$$

$$(\vec{r}_r\times\vec{r}_\theta)_3=r>0.\quad \text{"}\pm\text{"}=\text{"}+\text{"}$$

$$=2r^2(2sc^2-cs^2+c^3)\,\hat{i}$$

$$-2r^2(s^3-sc^2+2sc^2)\,\hat{j}+r\hat{k}$$

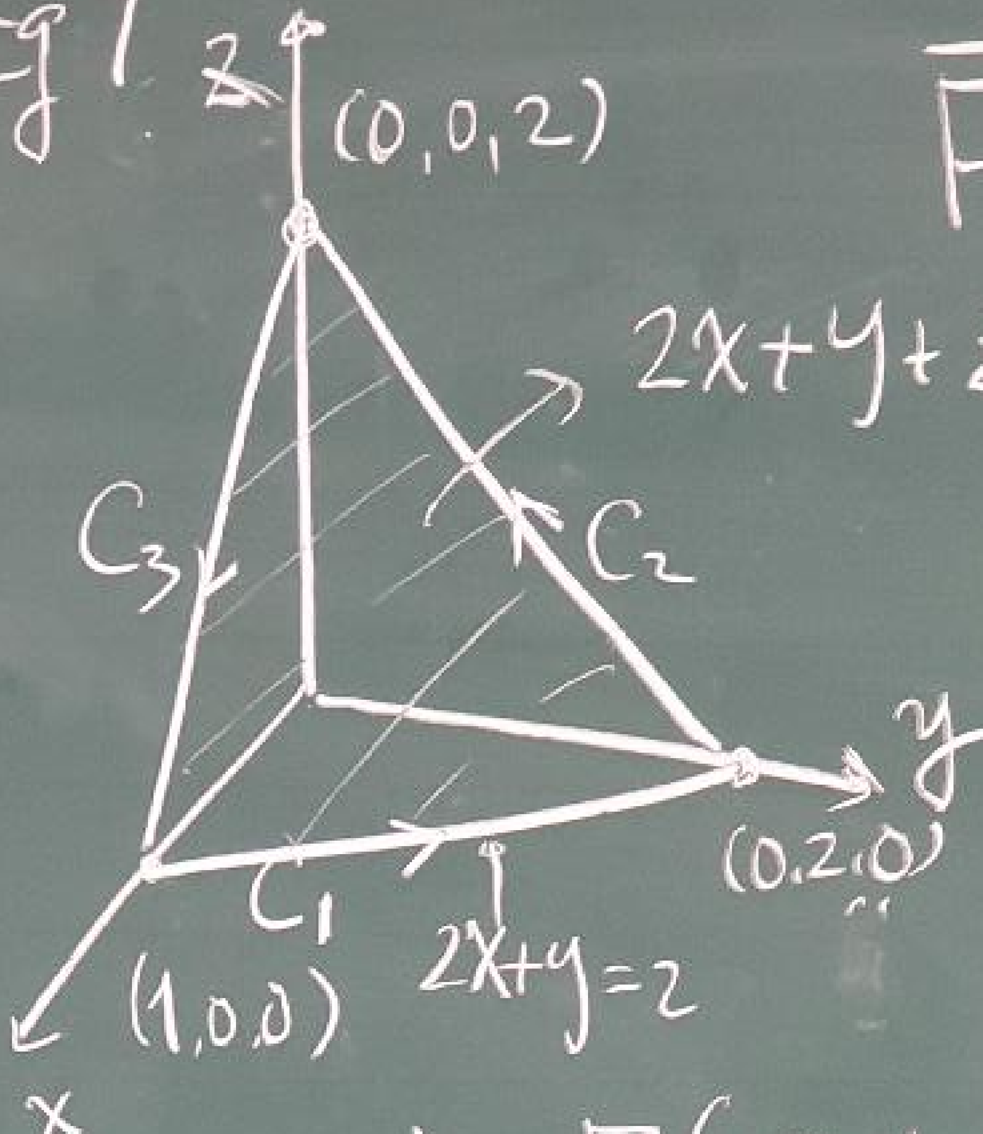
$$\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial_x & \partial_y & \partial_z\\ y & -x & x^2\end{vmatrix}=(0,\ \underset{-2rc}{\underset{\shortparallel}{-2x}},\ -2)$$

$$\iint_S=\int_{\theta=0}^{2\pi}\int_{r=0}^{1}4r^3(cs^3+sc^3)-2r\,dr\,d\theta$$

$$=\int_0^1(-2r)\,dr\int_0^{2\pi}d\theta=-2\pi$$

$$=\oint_{\partial S}\qquad\square$$

Eg 9.



$$\vec{F} = (xz,\ xy,\ 3xz)$$

$2x+y+z=2$, $\vec{n}$ = upward

$$\iint_S \nabla\times\vec{F}\cdot\vec{n}\,d\sigma = ?$$

$$\vec{n} = \pm\frac{\nabla(2x+y+z)}{|\ \cdot\ |} = +\frac{(2,1,1)}{\sqrt{6}}$$

$$d\sigma = \frac{dA}{\vec{n}\cdot\hat{z}} = \sqrt{6}\,dx\,dy$$

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ xz & xy & 3xz \end{vmatrix} = (0,\ x-3z,\ y)$$

$$\iint_S = \iint_{\triangle} \begin{pmatrix} 0 \\ x-3z \\ y \end{pmatrix}\cdot\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{2-2x} \underbrace{y+x-3(2-2x-y)}_{7x+4y-6}\,dy\,dx = -1$$

Method 2 $= \int_{C_1} + \int_{C_2} + \int_{C_3}$

$C_1: \vec{r_1}(y) = \left(1-\frac{y}{2}, y, 0\right)$

$C_2: \vec{r_2}(z) = (0, 2-z, z)$

$C_3: \vec{r_3}(x) = (x, 0, 2-2x)$

$= \int_0^2 \vec{F} \cdot \frac{d\vec{r_1}}{dy} dy + \int_{z=0}^2 \vec{F} \cdot \frac{d\vec{r_2}}{dz} dz$

$+ \int_0^1 \vec{F} \cdot \frac{d\vec{r_3}}{dx} dx$

Remark.

(1) $\nabla \times \nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \partial_x f & \partial_y f & \partial_z f \end{vmatrix} = \vec{0}$

(2) $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix}$

$\nabla \times \vec{F} = \vec{0} \Leftrightarrow \vec{F}$ satisfies component test

(3) $\nabla \times \vec{F} = \vec{0} \underset{?}{\Rightarrow} \vec{F} = \nabla f \left(\oint_C \vec{F} \cdot \vec{T}\, ds = 0 \quad \forall \text{ closed } C \right)$

Yes, if D is simply connected

Pf: If C is any simple closed curve,

$C \xrightarrow{\text{contracted}} \{x^*\}$, in the process

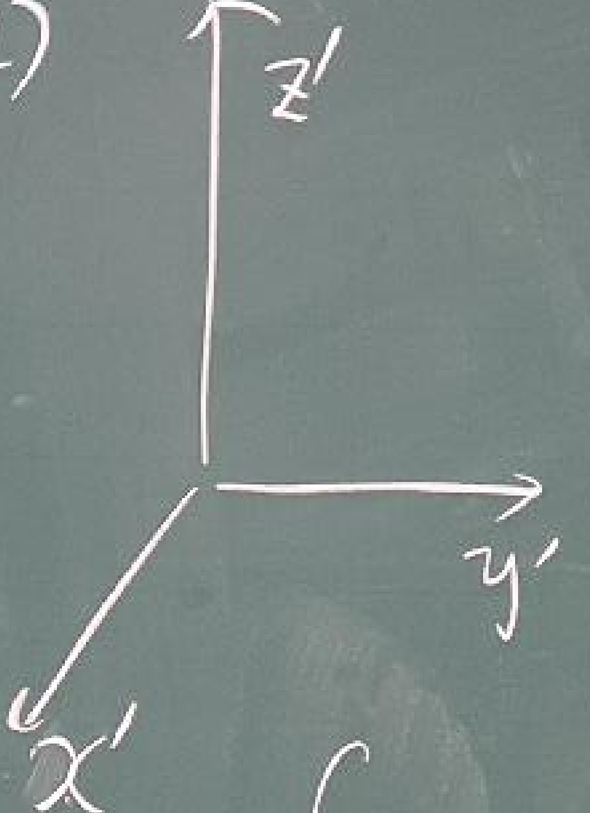
, we obtain $S \subseteq D$, $\partial S = C$

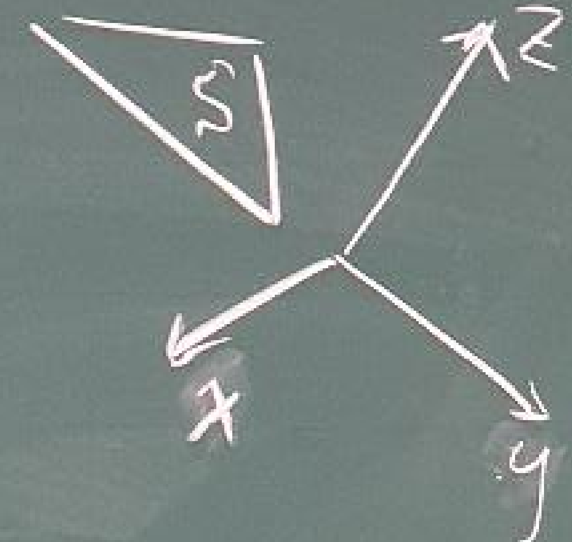
$\Rightarrow \oint_C \vec{F} \cdot \vec{T}\, ds = \iint_S \nabla \times \vec{F} \cdot \vec{n}\, d\sigma = 0$

From Green's Thm to Stokes' Thm

(1) Triangulation of surface

(2)





$$\oint_{\partial S} \vec{F}\cdot\vec{T}\,ds = \iint_S \underbrace{\nabla\times\vec{F}\cdot\vec{n}}_{\|}\,d\sigma$$

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

In (x', y', z') coord,

$$\vec{F} = F_1\hat{x} + F_2\hat{y} + F_3\hat{z}$$
$$= F_1'\hat{x}' + F_2'\hat{y}' + F_3'\hat{z}'$$

(Similarly for $\vec{T}$ and $\vec{n}$)

$$\begin{vmatrix} \hat{x}' & \hat{y}' & \hat{z}' \\ \partial_{x'} & \partial_{y'} & \partial_{z'} \\ F_1' & F_2' & F_3' \end{vmatrix} \overset{?}{=} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix} \Rightarrow \underline{\text{Yes}}.$$