If $S = \{ z = f(x,y),\ (x,y) \in R \subseteq \mathbb{R}^2 \}$

$x = x,\ \ y = y,\ \ z = f(x,y),\ (x,y) \in R$

$$\begin{pmatrix} x(u,v) = u,\ y(u,v) = v, \\ z(u,v) = f(x(u,v), y(u,v)),\ (u,v) \in R \end{pmatrix}$$

$$d\sigma = |\vec{r}_x \times \vec{r}_y|\, dx\,dy$$

$$\overset{\text{or}}{=} \frac{|\nabla F|}{|\nabla F \cdot \hat{z}|}\, dx\,dy$$

with $F(x,y,z) = f(x,y) - z$

$$= \sqrt{1 + f_x^2 + f_y^2}\, dx\,dy$$

Eg: $S = \{ z = \sqrt{x^2 + y^2},\ 0 \le z \le 1 \}$

Sol: $0 \le z \le 1 \Rightarrow x^2 + y^2 \le 1$

$$\iint_S d\sigma = \iint_R \sqrt{1 + z_x^2 + z_y^2}\, dA$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sqrt{2}\, dx\,dy = \int_0^{2\pi} \int_0^1 \sqrt{2}\, r\,dr\,d\theta$$

Eg $S=\{x^2+y^2+z^2=a^2\}$

$$\vec{r}(\theta,\phi)=(aS_\phi C_\theta,\ aS_\phi S_\theta,\ aC_\phi)$$

$$0\le\theta\le 2\pi,\quad 0\le\phi\le\pi$$

$$\vec{r}_\theta=(-aS_\phi S_\theta,\ aS_\phi C_\theta,\ 0)$$

$$\vec{r}_\phi=(aC_\phi C_\theta,\ aC_\phi S_\theta,\ -aS_\phi)$$

$$\vec{r}_\theta\perp\vec{r}_\phi \Rightarrow |\vec{r}_\theta\times\vec{r}_\theta|=|\vec{r}_\theta|\,|\vec{r}_\phi|$$

$$= a\sin\phi\cdot a = a^2\sin\phi$$

$$d\sigma = a^2\sin\phi\,d\phi\,d\theta$$

Eg $S=\{x^2+(y-3)^2=9,\ 0\le z\le 5\}$

Sol: $x(\theta,z)=3\cos\theta,\ y(\theta,z)=3+3\sin\theta$

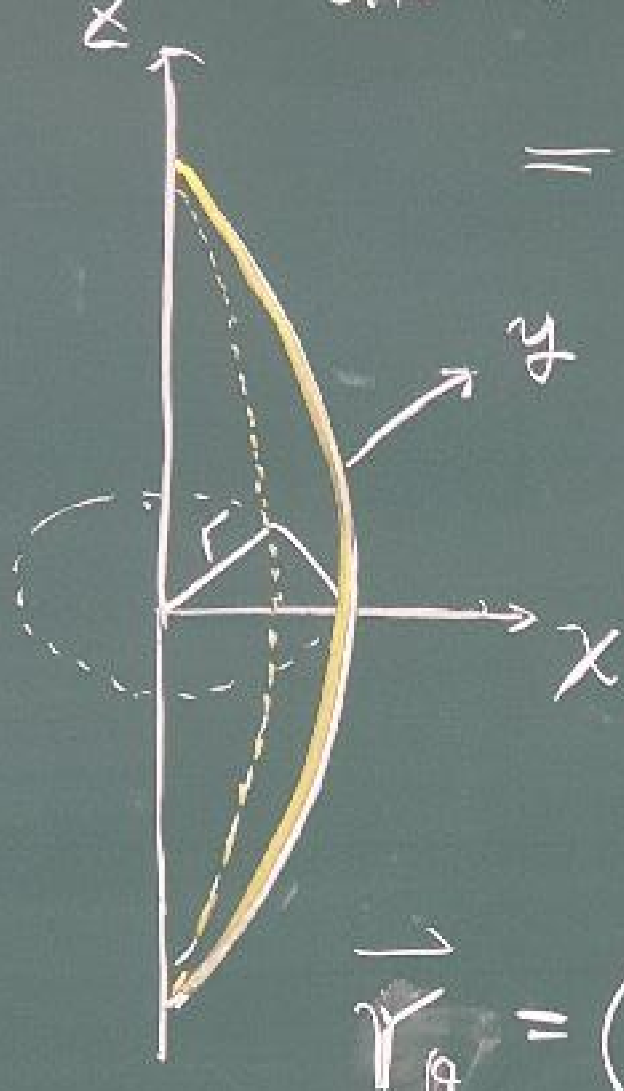
$$z(\theta,z)=z$$

$$d\sigma=|\vec{r}_\theta\times\vec{r}_z|\,dz\,d\theta \overset{\vec{r}_\theta\perp\vec{r}_z}{=\!=} |\vec{r}_\theta|\,|\vec{r}_z|\,dz\,d\theta.$$

Eg S = Surface of revolution

of $\left\{ x = \cos z,\ \frac{-\pi}{2} \le z \le \frac{\pi}{2} \right\}$

around z-axis.



$$= \left\{ r(\theta, z) = \cos z,\ \begin{matrix} \frac{-\pi}{2} \le z \le \frac{\pi}{2} \\ 0 \le \theta \le 2\pi \end{matrix} \right\}$$

$$\begin{cases} x(\theta, z) = r C_\theta = C_z C_\theta \\ y(\theta, z) = r S_\theta = C_z S_\theta \\ z(\theta, z) = z \end{cases}$$

$$\vec{r}_\theta = (-C_z S_\theta,\ C_z C_\theta,\ 0)$$

$$\vec{r}_z = (-S_z C_\theta,\ -S_z S_\theta,\ 1)$$

$$\vec{r}_\theta \perp \vec{r}_z \Rightarrow \left| \vec{r}_\theta \times \vec{r}_z \right| = C_z \sqrt{1 + S_z^2}$$

$$d\sigma = \cos z \sqrt{1 + \sin^2 z}\, dz\, d\theta$$

Eg $\displaystyle\iint_{\substack{z=\sqrt{x^2+y^2}\\ 0\le z\le 1}} x^2\, d\sigma = ?$

$$= \iint_{x^2+y^2\le 1} x^2\sqrt{2}\, dA$$

$$= \int_0^{2\pi}\int_{r=0}^{1} \sqrt{2}\, r^2\cos^2\theta\, r\, dr\, d\theta$$

$$= \left(\int_0^{2\pi}\cos^2\theta\, d\theta\right)\left(\int_0^1 \sqrt{2}\, r^3\, dr\right)$$

$$= \left(2\pi\cdot\frac{1}{2}\right)\cdot\frac{\sqrt{2}}{4}.$$

Eg $S = \{ z = \frac{y^2}{2},\ (x,y) \in R \}$

$R = \{ (x,y),\ x \geq 0,\ y \geq 0,\ x+y \leq 1 \}$

$$\iint_S \sqrt{x(1+2z)}\, d\sigma = ?$$

Sol: $d\sigma = \sqrt{1+z_x^2+z_y^2}\, dx\, dy$

$= \sqrt{1+y^2}\, dx\, dy$

$$\text{Ans} = \iint_R \sqrt{x(1+y^2)}\sqrt{1+y^2}\, dx\, dy$$

$$= \int_0^1 \int_{x=0}^{1-y} \sqrt{x}(1+y^2)\, dx\, dy$$

$$= \int_0^1 \int_{y=0}^{1-x} \sqrt{x}(1+y^2)\, dy\, dx$$

Flux over a surface $\iint_S \vec{F}\cdot\vec{n}\, d\sigma$

<u>Def</u>: S: a smooth surface.

S is orientable if there exists a continuous unit normal vector field on S.

How to compute flux over an orientable surface S?

<u>Case 1</u>: $S = \{\vec{r}(u,v),\ (u,v)\in R \subseteq \mathbb{R}^2\}$

$\vec{n} \perp \vec{r_u},\ \vec{n} \perp \vec{r_v}$

$$\vec{n} = \pm \frac{\vec{r_u}\times\vec{r_v}}{|\vec{r_u}\times\vec{r_v}|},\quad d\sigma = |\vec{r_u}\times\vec{r_v}|\,du\,dv$$

$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \pm\iint_R \vec{F}(\vec{r}(u,v))\cdot \vec{r_u}\times\vec{r_v}\, du\,dv$$

Case 2. $S = \{(x,y,z),\ g(x,y,z) = 0\}$

$$\vec{n} = \pm \frac{\nabla g}{|\nabla g|}$$

If $g(x,y,z)$ >0 outside S
$=0$ on S
<0 inside S

$$\Rightarrow \frac{\nabla g}{|\nabla g|} = \text{~~outward~~ outward unit normal}$$

Case 3. $S = \{z = f(x,y),\ (x,y) \in R\}$

$$\vec{r} = (x,\ y,\ f(x,y))$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = (-f_x, -f_y, 1)$$

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \pm \iint_R \begin{pmatrix} F_1(x,y,f(x,y)) \\ F_2(x,y,f(x,y)) \\ F_3(x,y,f(x,y)) \end{pmatrix} \cdot \begin{pmatrix} -f_x \\ -f_y \\ 1 \end{pmatrix} dx\,dy$$

upward

$\pm$: upward / downward normal

downward