Stokes' Thm

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma = \int_{\partial S} \vec{F} \cdot \vec{T}\, ds$$

Green's Thm (tangential form)

$S = R \subseteq \mathbb{R}^2$

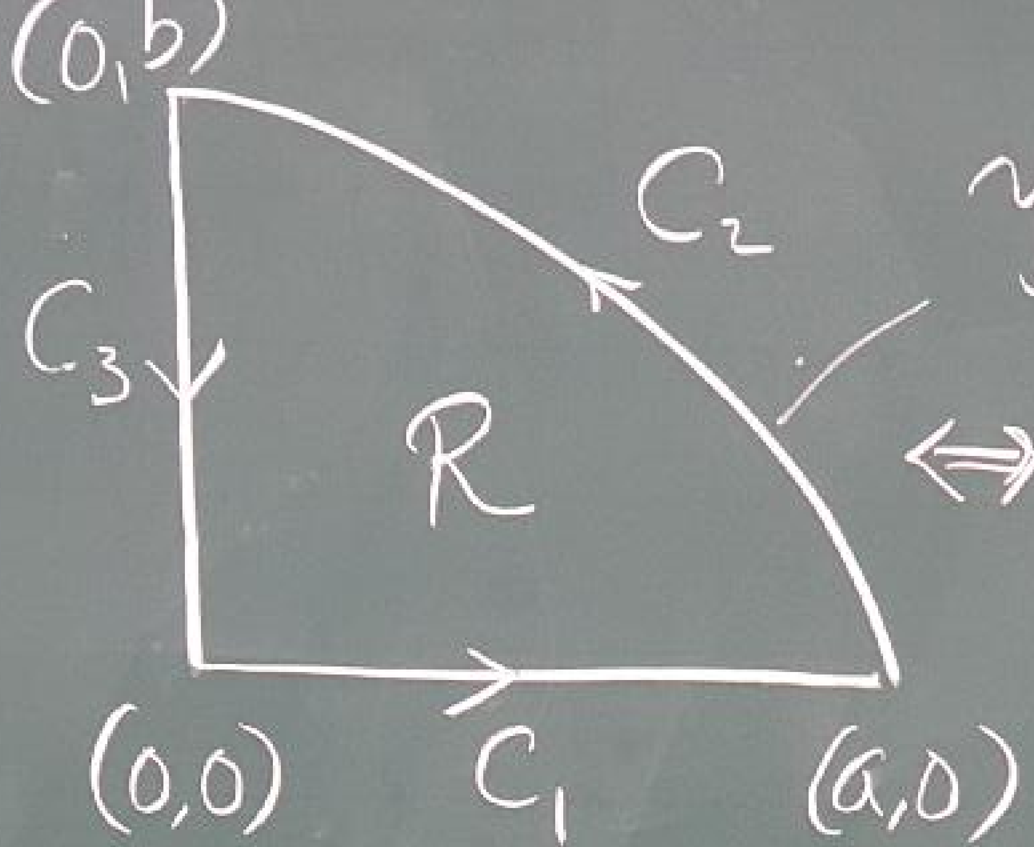
$\vec{F} = (M(x,y), N(x,y), 0)$

$\vec{n} = (0,0,1) \Longleftrightarrow \int_{\partial S} = \oint_{\partial S}$

$$\iint_R \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ M & N & 0 \end{vmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dA = \iint_R \begin{vmatrix} 0 & 0 & 1 \\ \partial_x & \partial_y & 0 \\ M & N & 0 \end{vmatrix} dA$$

$$(G_T): \iint_R (N_x - M_y)\, dA = \oint_{\partial R} M\,dx + N\,dy$$

Pf of G_T on a special region



$y = f(x),\ 0 \le x \le a$

$\Leftrightarrow x = g(y),\ 0 \le y \le b$

Next we check (G_T) on R

with $\vec{F} = (0, N)$

$$\iint_R N_x \, dA = \int_0^b \int_{x=0}^{g(y)} N_x(x,y)\, dx\, dy$$

$$= \underbrace{\int_0^b N(g(y), y)\, dy}_{\mathrm{II}^N} + \underbrace{\int_0^b -N(0,y)\, dy}_{\mathrm{III}}$$

$$\int_{C_3} (0,N)\cdot\vec{T}\,ds$$

parametrization

$$x(t)=0,\quad y(t)=b-t$$
$$0\le t\le b$$

$$\vec{T}\,ds=\begin{pmatrix}x'(t)\\ y'(t)\end{pmatrix}dt$$

$$\int_{C_3} = \int_{t=0}^{b} -N(0,b-t)\;dt$$

see remark below

$$y=b-t$$

$$=\int_{y=b}^{0} N(0,y)\,dy$$

$$=\int_{0}^{b} -N(0,y)\,dy$$

$$= \mathrm{III}$$

$$\begin{pmatrix}0\\ N\end{pmatrix}\cdot\begin{pmatrix}x'(t)\\ y'(t)\end{pmatrix}=\begin{pmatrix}0\\ N\end{pmatrix}\cdot\begin{pmatrix}0\\ -1\end{pmatrix}=-N$$

$$\int_{C_2} \vec{F}\cdot\vec{T}\,ds$$

Let $y=t$, $x=g(t)$

$$= \int_{t=0}^{b} \begin{pmatrix}0\\N\end{pmatrix}\begin{pmatrix}x'\\y'\end{pmatrix} dt$$

$$= \int_0^b N(g(t),t)\,dt = \mathrm{II}^N$$

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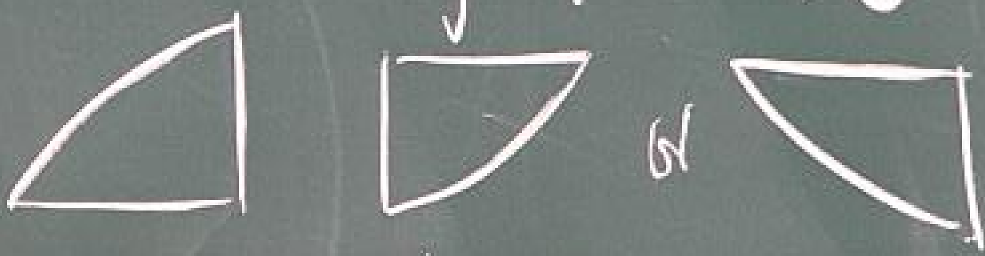
$\therefore \vec{F} = \begin{pmatrix}0\\N\end{pmatrix}$ on R checked.

Similarly for $\vec{F} = \begin{pmatrix}M\\0\end{pmatrix}$

(homework)

Similarly for $R =$



Also for $(G_N$

Eg $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) = (M, N)$

$C: \frac{x^2}{4} + y^2 = 1$, $\oint_C \vec{F} \cdot \vec{T}\, ds = ?$

Sol. Method 1: $\left(R: \frac{x^2}{4} + y^2 \le 1\right)$ ✗

$\left(R: 0 < \frac{x^2}{4} + y^2 \le 1\right)$

$N_x - M_y = 0$

$G_T \Rightarrow Ans = \iint_R 0\, dA = 0$

Method 2:

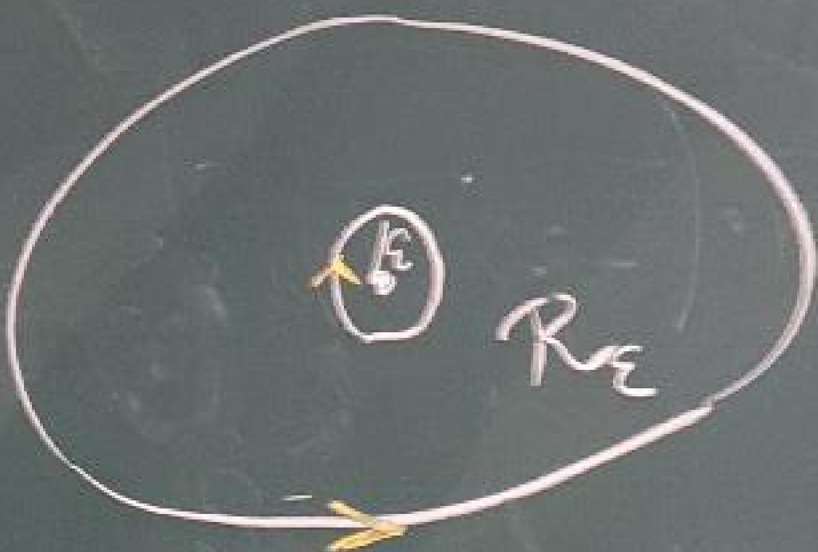
$x(t) = 2\cos t,\ y(t) = \sin t,\ 0 \le t \le 2\pi$

$$Ans = \int_{t=0}^{2\pi} \frac{\begin{pmatrix} -\sin t \\ 2\cos t \end{pmatrix} \cdot \begin{pmatrix} -2\sin t \\ \cos t \end{pmatrix}}{4\cos^2 t + \sin^2 t}\, dt$$

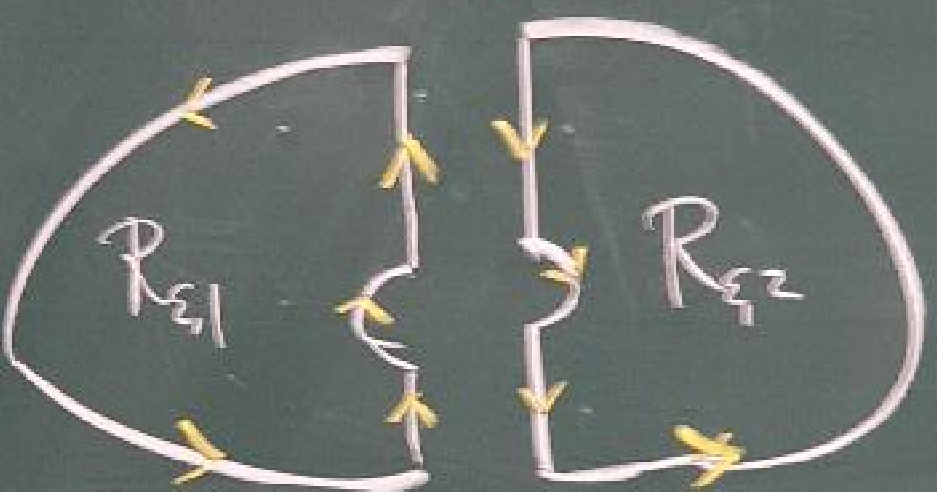
$$= \int_0^{2\pi} \frac{2}{4\cos^2 t + \sin^2 t}\, dt \neq 0$$

What went wrong?

Method 3: Apply (GT) to $R_\varepsilon = \left\{ \varepsilon^2 < \frac{x^2}{4} + y^2 < 1 \right\}$



$$R_\varepsilon = R_{\varepsilon,1} \cup R_{\varepsilon,2}$$



$$\oint_{\partial R_\varepsilon} = \oint_{\frac{x^2}{4}+y^2=1} + \oint_{x^2+y^2=\varepsilon^2}$$

$$= \oint_{\frac{x^2}{4}+y^2=1} - \oint_{x^2+y^2=\varepsilon^2}$$

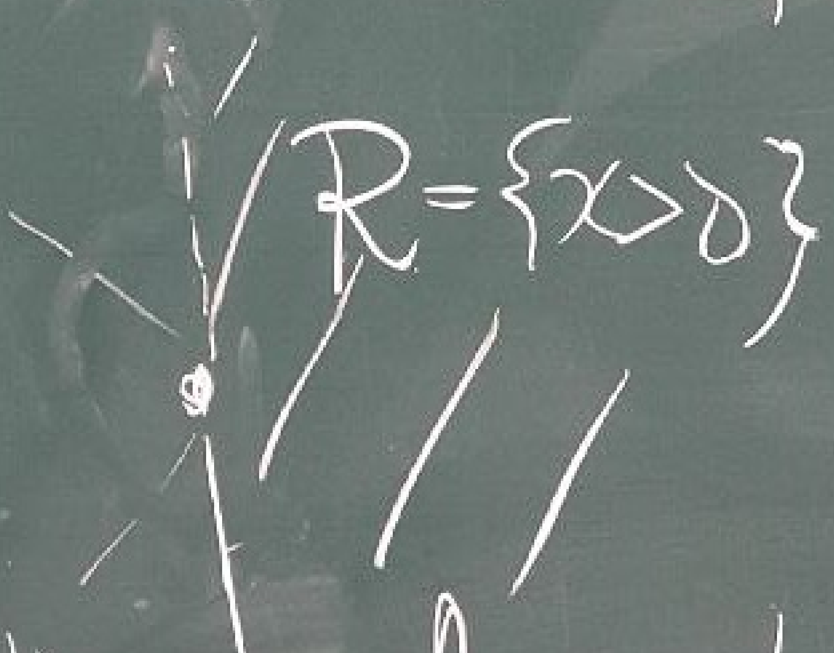
$$\iint_{R_\varepsilon} N_x - M_y \, dA = \oint_C \vec{F}\cdot\vec{T}\, ds - \oint_{x^2+y^2=\varepsilon^2} \vec{F}\cdot\vec{T}\, ds$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underset{2\pi}{\parallel}$$

$$\therefore \oint_C \vec{F}\cdot\vec{T}\, ds = \iint_{R_\varepsilon} 0 \, dA + \oint_{x^2+y^2=\varepsilon^2} \vec{F}\cdot\vec{T}\, ds = 2\pi$$

<u>Rmk</u> $\vec{F} = \dfrac{(-y,x)}{x^2+y^2} = \nabla \tan^{-1}\left(\dfrac{y}{x}\right) = \nabla\theta$

when $(x,y) \in \mathrm{I}, \mathrm{IV}$ (or $x>0$)



However, θ cannot be defined as a cont. funtion on $\mathbb{R}^2 \setminus \{(0,0)\}$

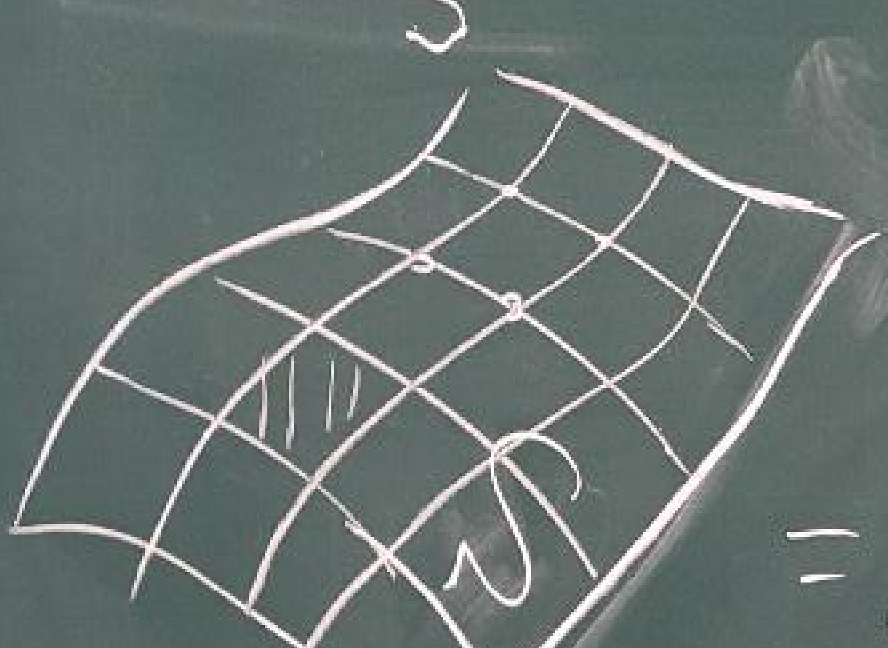
<u>Remark</u> If $\vec{F} = \frac{(-y,x)}{x^2+y^2}$

C: a simple closed curve

Then $\oint_C \vec{F}\cdot\vec{T}\,ds = \begin{cases} 0, & (0,0)\notin \text{interior} \\ 2\pi, & (0,0)\in \text{interior} \end{cases}$

Surface Area and Surface integrals

<u>Def</u> $\iint_S f(x,y,z)\,d\sigma$, S: a surface in $\mathbb{R}^3$



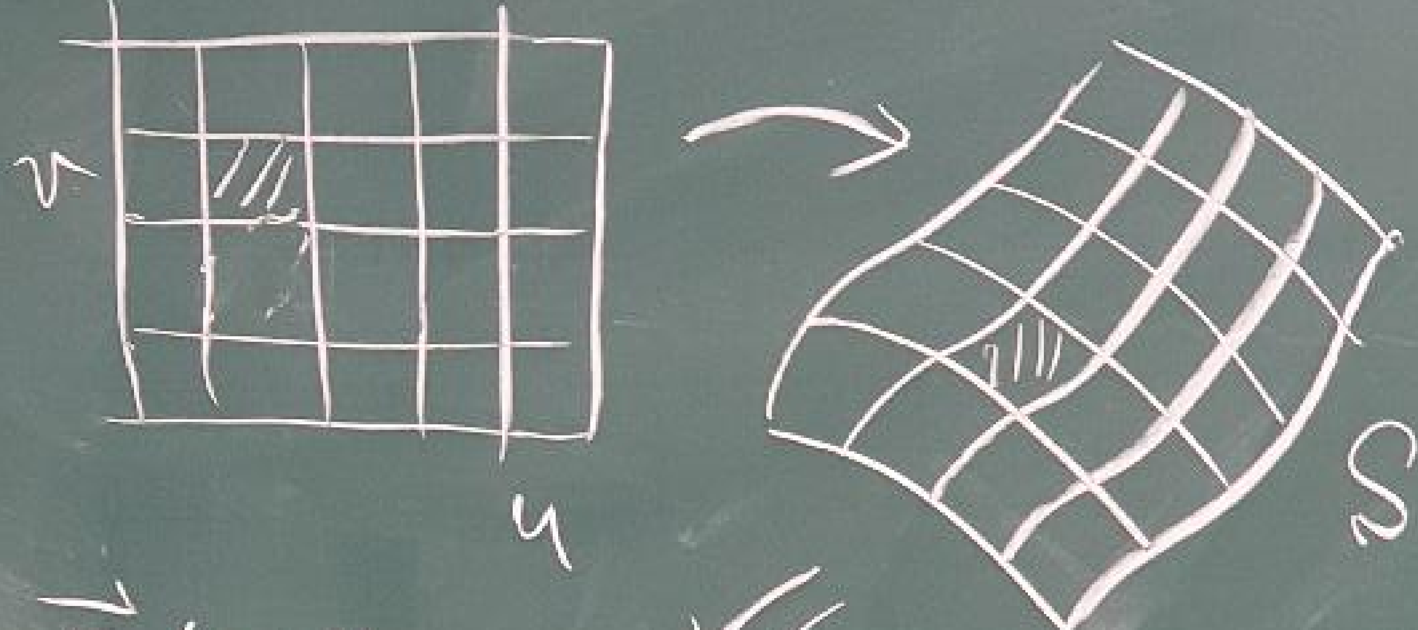
$\to$ Area $= \Delta A_k$

$= \lim_{\|P\|\to 0} \sum_{k=0}^{n} f(\tilde{x}_k, \tilde{y}_k, \tilde{z}_k)\,\Delta A_k$

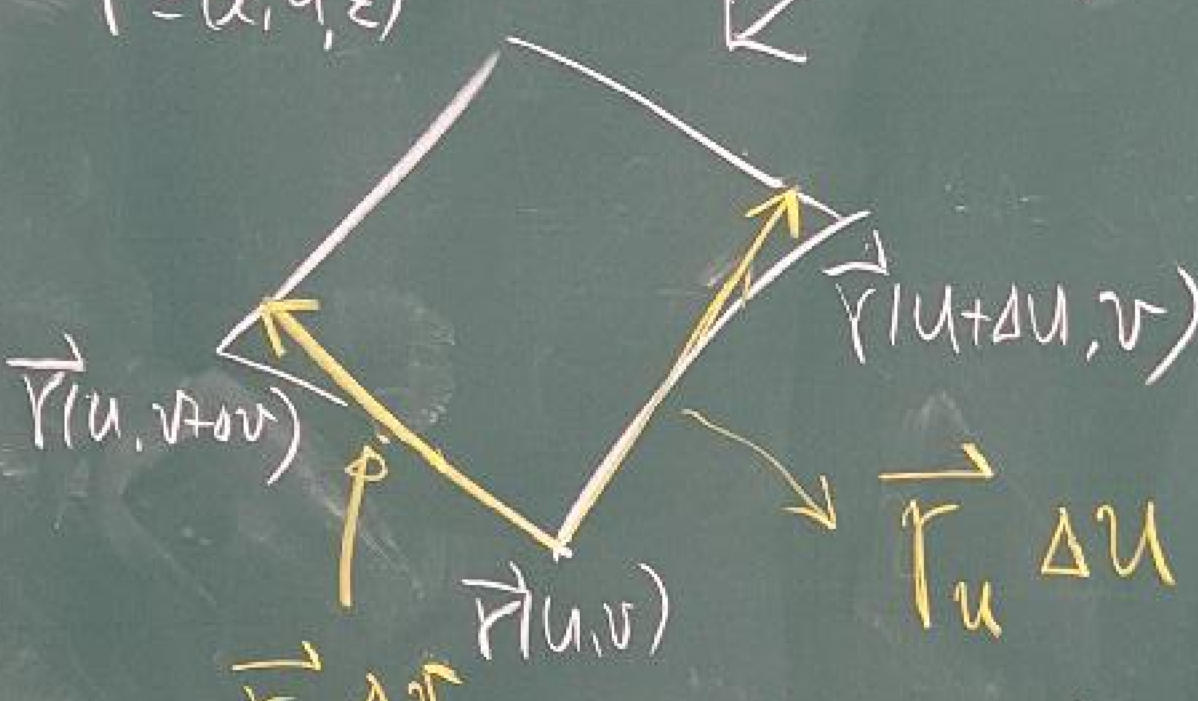
$(\tilde{x}_k, \tilde{y}_k, \tilde{z}_k) \in S_k$

How to Compute ΔA_k or $d\sigma$?

<u>Case 1</u> $S = \left\{ \begin{array}{c} (x(u,v), y(u,v), w(u,v)) \\ a \le u \le b,\ c \le v \le d \end{array} \right\}$



$\vec{r} = (x, y, z)$



$$\Delta A_k = \left| \vec{r}_u \times \vec{r}_v \right| \Delta u \Delta v$$

$$\therefore \iint_S f(x,y,z)\, d\sigma = \int_c^d \int_{u=a}^{b} f(\vec{r}(u,v)) \left| \vec{r}_u \times \vec{r}_v \right| du\, dv$$

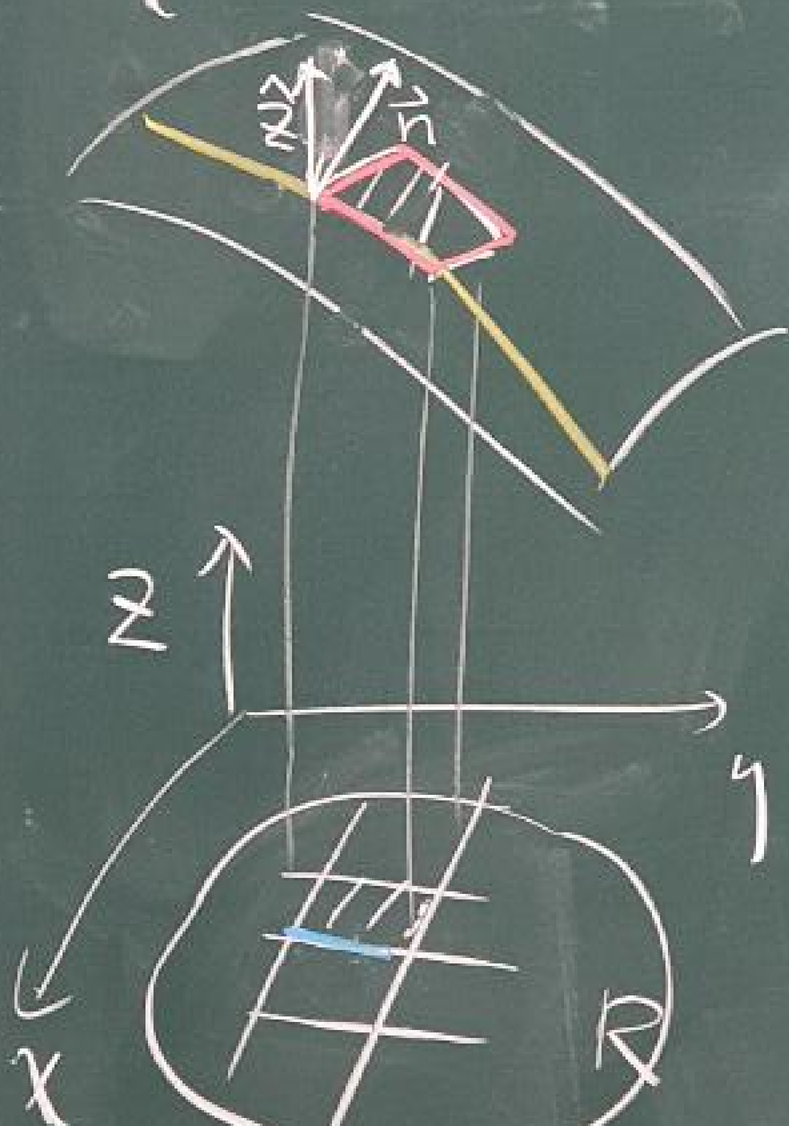
Case 2

$$S = \{ (x, y, g(x,y)),\ (x,y) \in R \}$$

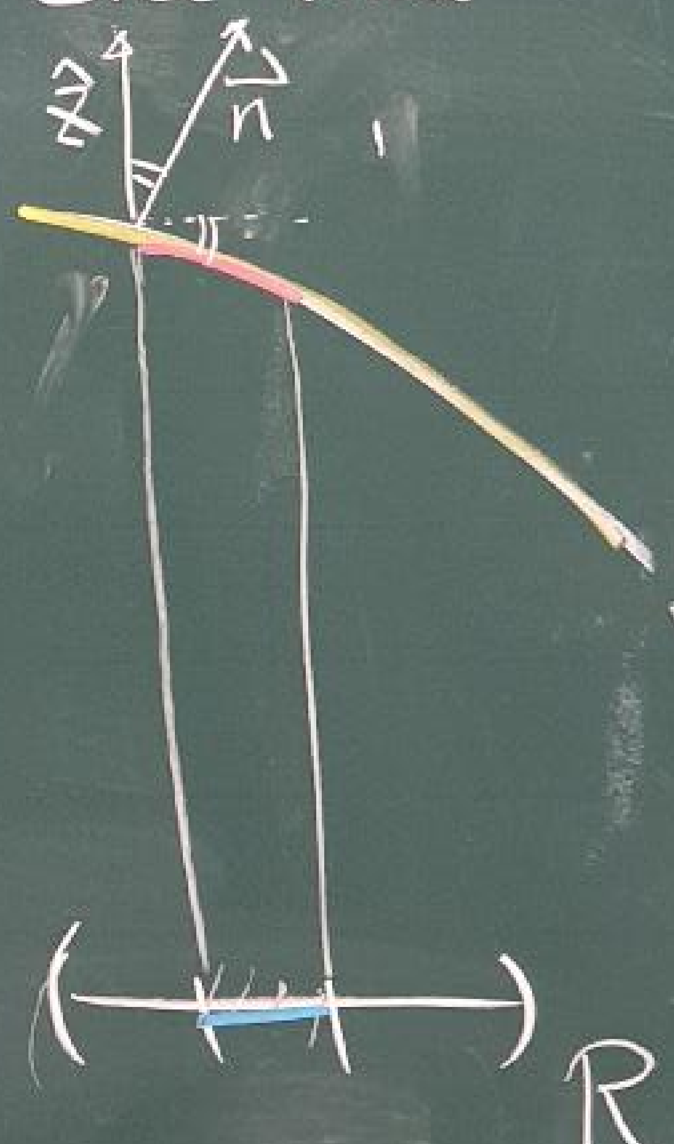
$$\left(\begin{array}{ll} \text{or } (x, g(x,z), z) & (x,z) \in R \\ \quad\ (g(y,z), y, z) & (y,z) \in R \end{array} \right)$$

in the form of

$$\{ F(x,y,z) = 0,\ (x,y) \in R \}$$



Side views



$$\frac{dA}{d\sigma} = |\cos\theta|,\quad \theta = \angle(\vec{n}, \vec{z})$$

$$|\cos\theta| = |\vec{n}\cdot\hat{z}|$$

From the last equation on previous page,

$$S: F(x,y,z)=0 \qquad d\sigma = dA/|\cos\theta|$$

$$\Rightarrow \vec{n} = \nabla F/|\nabla F|$$

$$\therefore \iint_S f(x,y,z)\,d\sigma$$

$$= \iint_R f(x,y,z)\left|\frac{|\nabla F|}{\nabla F\cdot\hat{p}}\right| dA$$

Here $\hat{p} = \hat{z}$ if $R \in \{x\text{-}y \text{ plane}\}$

$\hat{y}$ $\{x\text{-}z \text{ plane}\}$

$\hat{x}$ $\{y\text{-}z \text{ plane}\}$

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