Summary: If $\vec{F}$ is cont. C^1

in $D \subseteq \mathbb{R}^2$ or $\mathbb{R}^3$. Then

$\vec{F}$ is conservative in D

$\Leftrightarrow \oint_C \vec{F}\cdot\vec{T}\,ds = 0$, C: any closed curve in D.

$\Leftrightarrow \vec{F} = \nabla f$, f is diff. in D

$\Rightarrow \vec{F}$ satisfies the component ~~is exact~~ test

If D is simply connected

Then "$\Leftarrow$" also holds.

<u>Def</u> D is simply connected

if every closed curve in D

can be cont. contracted to a point

in D without leaving D.

Are they simply connected?

Eg

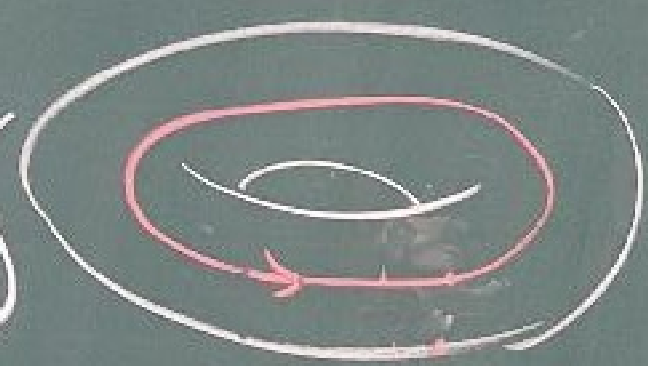
$\{x^2+y^2+z^2<4\}$, yes

$\{1<x^2+y^2+z^2<4\}$, yes

$\mathbb{R}^3\setminus\{(0,0,0)\}$, yes

$1<x^2+y^2<4$ in $\mathbb{R}^3$, no

$\mathbb{R}^3\setminus\{z \text{ axis}\}$, no



$(\sqrt{x^2+y^2}-2)^2+z^2<1$, no

$(\sqrt{x^2+y^2}-2)^2+z^2=1$, no

Eg $\vec{F} = \left(\underset{M}{\frac{-y}{x^2+y^2}}, \underset{N}{\frac{x}{x^2+y^2}}, 0 \right)$

It is easy to check that $M_y = N_x$

Note: $D = \mathbb{R}^2 \setminus \{(0,0)\}$ is not simply conneted.

C: $\vec{r}(t) = (r\cos t, r\sin t)$

$r > 0$ fixed. $0 \le t \le 2\pi$

$$\oint_C \vec{F} \cdot \vec{T}\, ds = ?$$

On C: $M = \frac{-r\sin t}{r^2}$, $N = \frac{r\cos t}{r^2}$

$$\int_0^{2\pi} \frac{1}{r}\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \cdot r \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} dt = 2\pi$$

Green's Thm.

Stokes' Theorem: $\vec{F}: D \subseteq \mathbb{R}^3 \to \mathbb{R}^3$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma = \int_{\partial S} \vec{F} \cdot \vec{T}\, ds$$

Divergence Thm

$$\iiint_D \nabla \cdot \vec{F}\, dV = \iint_{\partial D} \vec{F} \cdot \vec{n}\, d\sigma$$

S: a surface in D.

$d\sigma$: element in surface integral

$\vec{n}$: normal of S (or ∂D)

∂S, ∂D: boundary of S, D

If $\vec{F} = (M(x,y), N(x,y), 0)$

$S = \mathcal{R} \subseteq \mathbb{R}^2$, $d\sigma = dA$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial \\ M & N & 0 \end{vmatrix} = (N_x - M_y)\hat{k}$$

$\vec{n} = \hat{k}$, $\vec{T} \leftrightarrow \circlearrowleft$

Stokes Thm reduces to

Green's Thm (tangential form)

Let C be a simple closed curve

R = interior of C, If $\vec{F} = (M, N)$

$M(x,y), N(x,y)$ and their 1st derivatives

are all continuous in R. Then

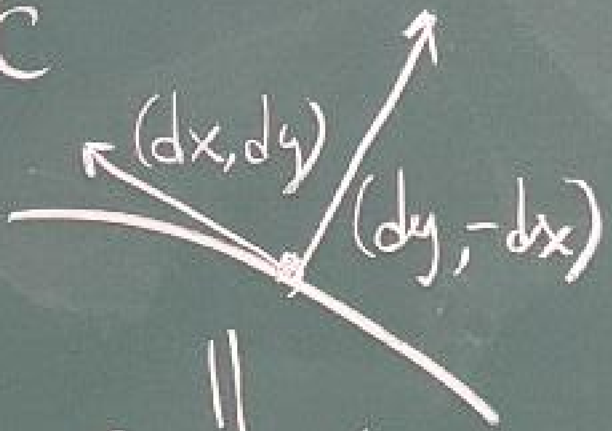
$$(G_T): \oint_C \vec{F} \cdot \vec{T}\, ds = \iint_R (N_x - M_y)\, dA$$

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \oint_C M\,dx + N\,dy$$

If we apply (G_T) to

$\vec{G} = (-N, M)$

$$\oint_C -N\,dx + M\,dy = \iint_R (M_x + N_y)\,dA$$



(G_N)

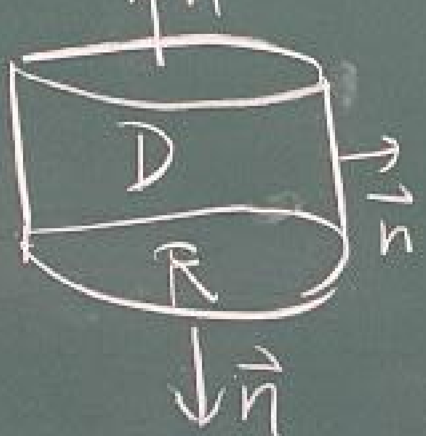
$$\| $$

$$\oint_C \begin{pmatrix} M \\ N \end{pmatrix} \cdot \begin{pmatrix} dy \\ -dx \end{pmatrix} = \oint_C \vec{F} \cdot \vec{n}\, ds$$

This is the special case of Divergence Thm with

$\vec{F} = (M(x,y), N(x,y), 0)$

$D = R \times (0,1)$



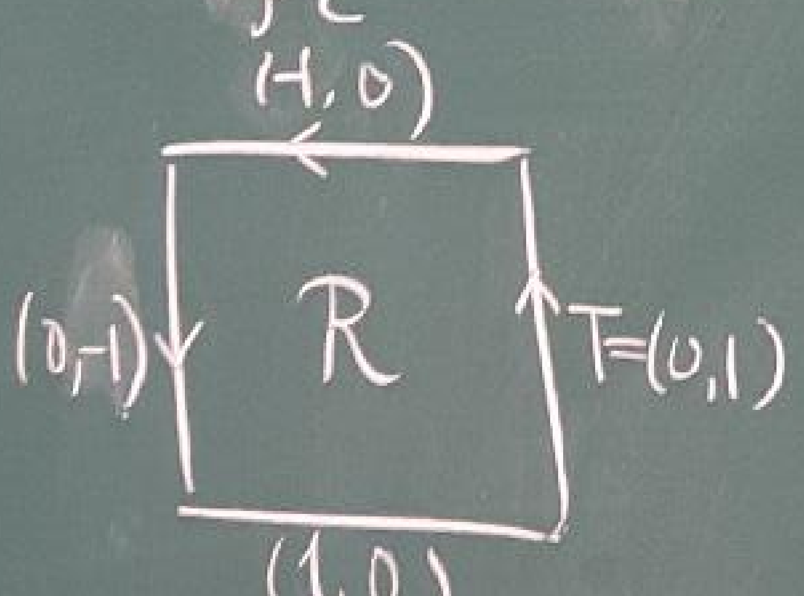
Green's Thm = F.T.C on R

Eg. $R = (a,b) \times (c,d)$

$$\iint_R N_x \, dA = \int_c^d \int_a^b N_x \, dx \, dy$$

$$= \int_c^d \big(N(b,y) - N(a,y)\big) \, dy$$

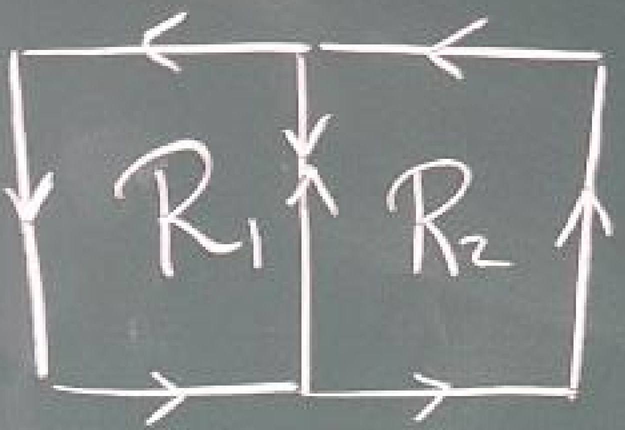
$$= \int_{y=c}^{d} \begin{pmatrix}0\\N\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\Bigg|_{x=b} dy + \int_{y=c}^{d} \begin{pmatrix}0\\N\end{pmatrix}\cdot\begin{pmatrix}0\\-1\end{pmatrix}\Bigg|_{x=a} dy$$



$$= \oint_{\partial R} \begin{pmatrix}0\\N\end{pmatrix} \cdot \vec{T} \, ds$$

Similarly, $-\iint_R M_y \, dA = \oint_{\partial R} \begin{pmatrix}M\\0\end{pmatrix} \cdot \vec{T} \, ds$

Remark



$$\iint_{R_1 \cup R_2} = \iint_{R_1} + \iint_{R_2}$$

$$\oint_{\partial R_1} \vec{F} \cdot \vec{T}\, ds + \oint_{\partial R_2} \vec{F} \cdot \vec{T}\, ds = \oint_{\partial (R_1 \cup R_2)} \vec{F} \cdot \vec{T}\, ds$$