Eg: $\vec{F} = (z, xy, -y^2)$

C: $\vec{r}(t) = (t^2, t, \sqrt{t}),\ 0 \le t \le 1$

$$\int_C \vec{F} \cdot \vec{T}\, ds = ?$$

<u>Sol</u>: $\vec{T}\, ds = \frac{d\vec{r}(t)}{dt} dt$

$$\frac{d\vec{r}(t)}{dt} = \left(2t, 1, \frac{1}{2\sqrt{t}}\right)$$

$$\text{Ans} = \int_{t=0}^{1} \begin{pmatrix} \sqrt{t} \\ t^3 \\ -t^2 \end{pmatrix} \cdot \begin{pmatrix} 2t \\ 1 \\ 1/2\sqrt{t} \end{pmatrix} dt$$

$$= \int_0^1 \left(2t^{\frac{3}{2}} + t^3 - \frac{t^{\frac{3}{2}}}{2}\right) dt$$

$$= \frac{1}{4} + \frac{3}{5}$$

Rm $\displaystyle\int_C \vec{F}\cdot\vec{T}\,ds$

$\displaystyle = \int_C \vec{F}\cdot\frac{d\vec{r}(t)}{dt}\,dt$

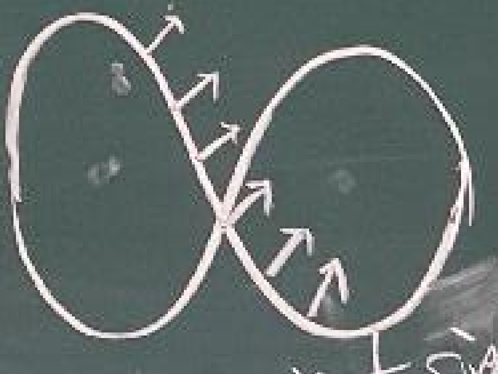
$\displaystyle = \int_C F_1\,dx + F_2\,dy + F_3\,dz$

Rm If $\vec{F}$ = force

then $\displaystyle\int_C \vec{F}\cdot\vec{T}\,ds$ = work

Rm If C is a simple closed curve, we use the notation $\displaystyle\oint_C \vec{F}\cdot\vec{T}\,ds$

simple



not simple

Related line integral

(3): $\int_C \vec{F}\cdot\vec{n}\,ds \quad \left(\oint_C \vec{F}\cdot\vec{n}\,ds\right)$

C: Simple closed curve in $D \subseteq \mathbb{R}^2$

$\vec{n}$ = outward unit normal.

$\vec{r}(t) = (x(t), y(t))$ = counter-clockwise parametrization of C

$$\vec{T} = \frac{d\vec{r}(t)}{dt} \Big/ \left|\frac{d\vec{r}(t)}{dt}\right|$$

$$\vec{T} = (t_1, t_2) \Rightarrow \vec{n} = (t_2, -t_1)$$

$$= \frac{(x'(t), y'(t))}{\sqrt{x'^2 + y'^2}} \qquad = \frac{(y'(t), -x'(t))}{\sqrt{x'^2 + y'^2}}$$

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \int_{t=a}^{b} \begin{pmatrix} F_1(x(t), y(t)) \\ F_2(x(t), y(t)) \end{pmatrix} \cdot \begin{pmatrix} y'(t) \\ -x'(t) \end{pmatrix} dt$$

Note: $ds = \sqrt{x'^2 + y'^2}\,dt$

Eg: $\vec{F} = (x-y, y)$

$C: x^2+y^2=1$, $\int_C \vec{F}\cdot\vec{n}\,ds = ?$

Sol: $x(t) = \cos t$

$y(t) = \sin t$ $\quad 0 \le t \le 2\pi$

$$\frac{d\vec{r}}{dt} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$

$$\vec{T} = \frac{d\vec{r}}{dt}, \quad \vec{n} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

$$\text{Ans} = \int_0^{2\pi} \begin{pmatrix} \cos t - \sin t \\ \sin t \end{pmatrix} \cdot \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} dt$$

$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t - \cos t \sin t)\, dt$$

$$= 2\pi$$

Thm If f is diff. and ∇f is cont. in D, $C \subseteq D$ is <u>any</u> smooth curve from $A \in D$ to $B \in D$

Then $\int_C \nabla f \cdot \vec{T}\, ds = f(B) - f(A)$

(i.e. $\int_A^B \nabla f \cdot \vec{T}\, ds$ is path-indep.)

pf: Let $\vec{r}(t)$, $0 \le t \le 1$ be a parametrization of C, $\vec{r}(0) = A$, $\vec{r}(1) = B$

Let $g(t) = f(x(t), y(t))$, $0 \le t \le 1$

$$\Rightarrow g(1) - g(0) = \int_0^1 g'(t)\, dt \quad (\text{F.T.C})$$

$$\| $$

$$f(B) - f(A) = \int_0^1 \begin{pmatrix} f_x \\ f_y \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} dt = \int_C \vec{F} \cdot \vec{T}\, ds$$

Def: $\vec{F}$ is conservative in D if for any $C \subseteq D$, $\int_C \vec{F}\cdot\vec{T}\,ds$ only depends on the end points of C.

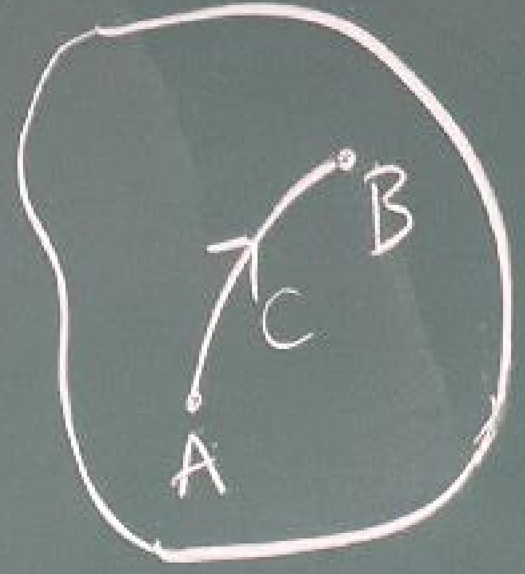
Prev Thm: $\vec{F} = \nabla f \Rightarrow \vec{F}$ is cons.
conservative

Next Thm: " $\Longleftrightarrow$ "

Thm: Let D be an open connected domain, then
" $\vec{F} = \nabla f$ for some function f in D (f is called potential of $\vec{F}$)"
$\Leftrightarrow \vec{F}$ is conservative in D

Pf: " $\Rightarrow$ " = prev Thm

"$\Leftarrow$"



Take any $A \in D$

and define $f(A) = 0$

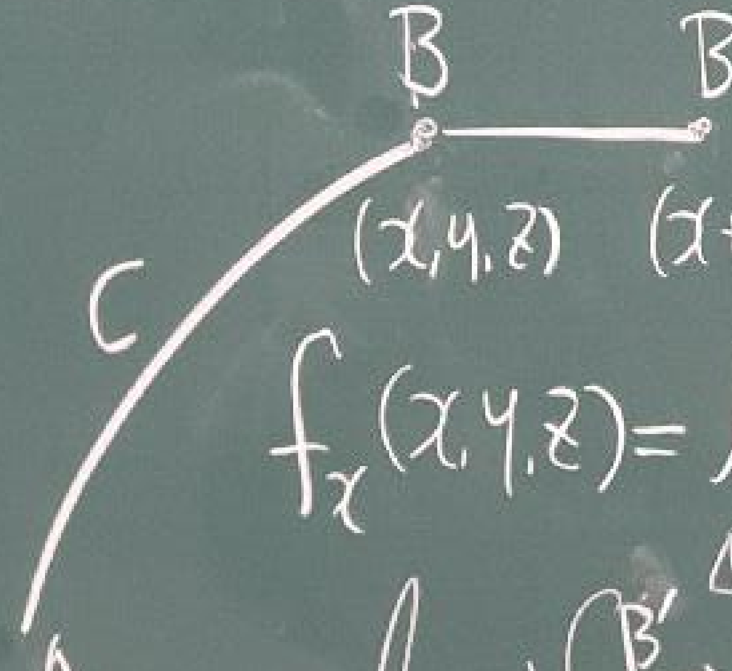
Then for any $B \in D$,

we define $f(B) = \int_C \vec{F} \cdot \vec{T}\, ds$

where C is any curve from A to B

It remains to show that $\nabla f = \vec{F}$

Eg: $f_x(x,y,z) \overset{?}{=} F_1(x,y,z)$



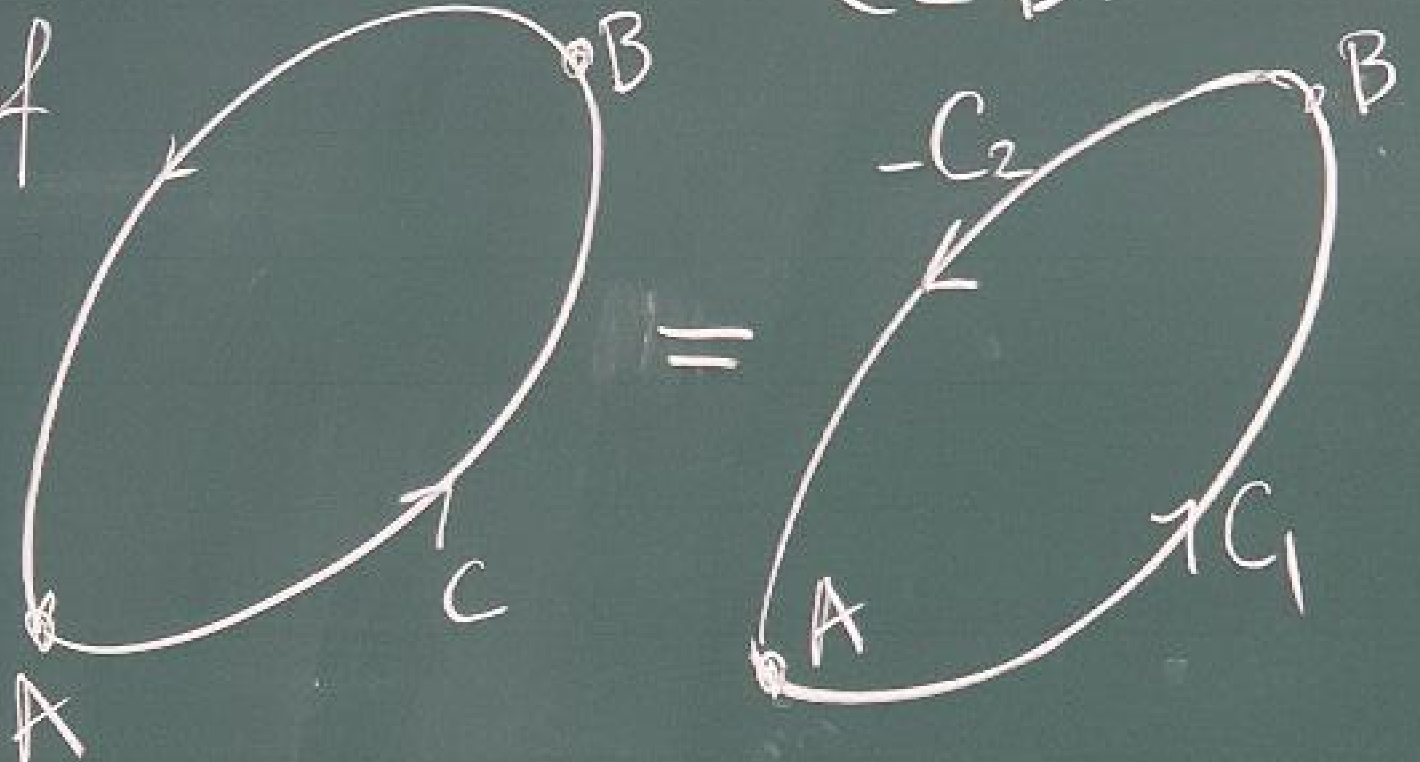
$$f_x(x,y,z) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_B^{B'} \vec{F} \cdot \underset{\substack{\| \\ (1,0,0)}}{\vec{T}}\, ds = \lim_{\Delta x \to 0} \frac{\int_{t=x}^{x+\Delta x} F_1(t,y,z)\, dt}{\Delta x} = F_1(x,y,z)$$

Rm: $\vec{F}$ is conservative in D

$\Longleftrightarrow \int_C \vec{F}\cdot\vec{T}\,ds = 0$ for any closed curve $C \subseteq D$

pf



$$\int_C = \int_{C_1} + \int_{-C_2} = \int_{C_1} - \int_{C_2}$$

$$= 0$$

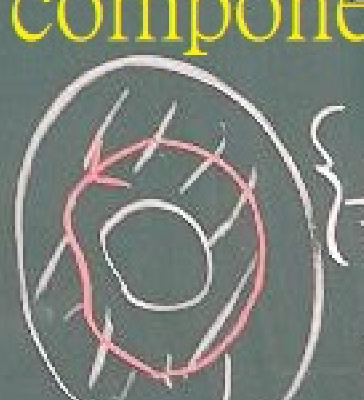
Def. $\vec{F}(x,y) = (M(x,y), N(x,y))$

$\left(\vec{F}(x,y,z) = (M(x,y,z), N(x,y,z), P(x,y,z))\right)$

~~is exact~~ satisfies the component test if $M_y = N_x$ $\left(\begin{cases} M_y = N_x \\ N_z = P_y \\ P_x = M_z \end{cases}\right)$

Cor: $\vec{F} = \nabla f \Rightarrow \vec{F}$ ~~is exact~~ satisfies the component test

Rm If D is simply connected then $\vec{F} = \nabla f \Leftrightarrow \vec{F}$ ~~is exact~~ satisfies the component test

Eg. in 2D:  $\{r<1\}$ Yes  $\{\frac{1}{2}<r<1\}$ No.

Simply con?
(simply connected?)

In 3D: $\{\rho<1\}$: Yes, $\{0<\rho<1\}$: Yes, $\{0<r<1\}$: No

Eg Show that

$$\vec{F} = \left(e^{x}\cos y + yz,\ xz - e^{x}\sin y,\ xy + z\right)$$

is conservative by finding its potential.

Sol: Check $\vec{F}$ ~~is exact~~ satisfies the component test:

$$M_y = -e^{x}\sin y + z = N_x$$

$$N_z = x = P_y$$

$$P_x = y = M_z$$

$\vec{F}$ is defined on $\mathbb{R}^3 \Rightarrow D = \mathbb{R}^3$ (Simply connected)

$\Rightarrow f$ exists.

How to find f ?

$$f_x = e^x \cos y + yz$$

(w.r.t. = with respect to)

$$\Rightarrow f(x,y,z) = e^x \cos y + xyz + g_1(y,z)$$

(integrate w.r.t. x)

Similarly

$$= xyz + e^x \cos y + g_2(x,z)$$

(integrate w.r.t. y)

$$= xyz + \frac{z^2}{2} + g_3(x,y)$$

(integrate w.r.t. z)

$$= e^x \cos y + xyz + \frac{z^2}{2} + C$$

Check: $f_x = e^x \cos y + yz$

$$f_y = -e^x \sin y + xz$$

$$f_z = xy + z$$