Remark

$$(\rho, \phi, \theta) \longleftrightarrow (r, \theta, z)$$

$$d\rho\, d\phi\, d\theta \longleftrightarrow dr\, dz\, d\theta$$

$$(\rho, \phi) \longleftrightarrow (z, r)$$

$$\rho\cos\phi = z, \quad \rho\sin\varphi = r$$

$$(R, \theta) \longrightarrow (X, Y)$$

Example: $0 \leq \rho \leq 2\sin\phi$

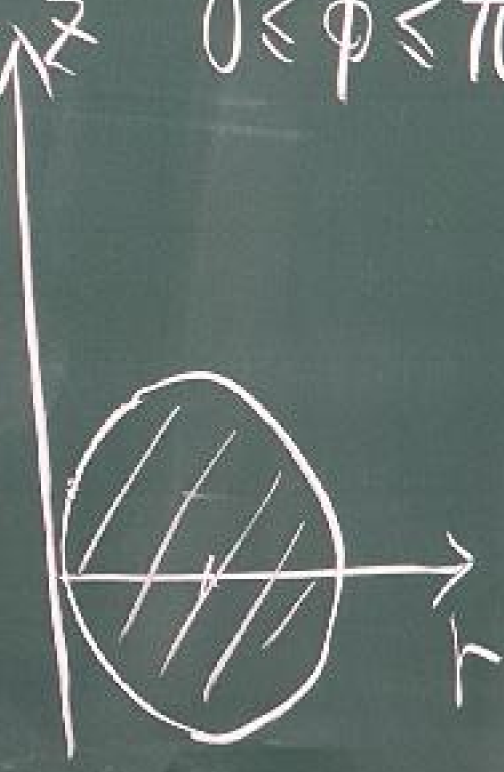
$0 \leq \phi \leq \pi$

$$\rho = 2\sin\phi$$

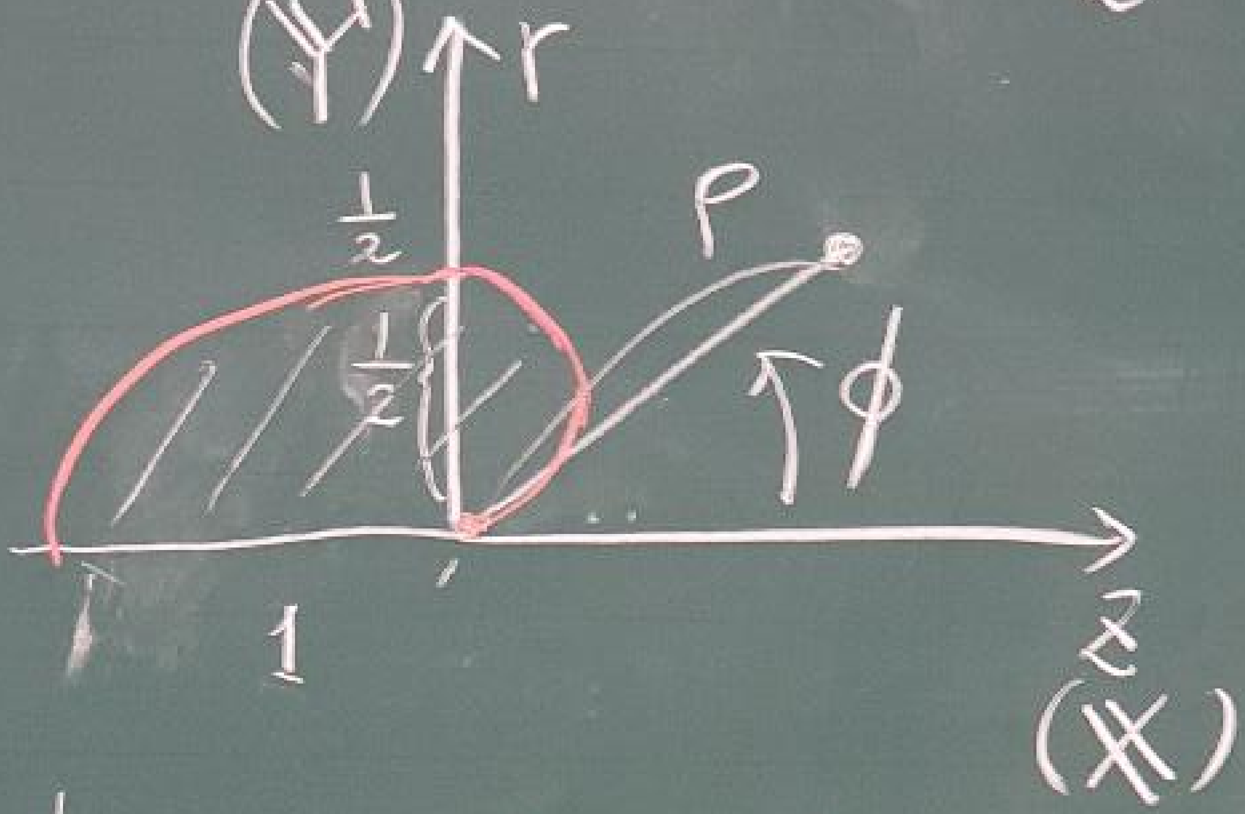
$$\Rightarrow \rho^2 = 2\rho\sin\phi$$

$$\Rightarrow r^2 + z^2 = 2r$$

$$\Rightarrow (r-1)^2 + z^2 = 1$$



Example. $0 \leq \rho \leq \dfrac{1-\cos\phi}{2}$



Line integrals.

Notations.

$$f : \mathbb{R}^3 \longmapsto \mathbb{R}$$
$$(x, y, z) \longrightarrow f$$

$$\vec{F} : \mathbb{R}^3 \longmapsto \mathbb{R}^3$$
$$(x, y, z) \longrightarrow \begin{pmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ F_3(x,y,z) \end{pmatrix}$$

Gradient of f: ∇f

curl of $\vec{F}$: $\nabla \times \vec{F}$ (curl $\vec{F}$)

divergence of $\vec{F}$: $\nabla \cdot \vec{F}$ (div $\vec{F}$)

$$\nabla \longleftrightarrow (\partial_x, \partial_y, \partial_z)$$

$$\nabla f \overset{\text{def}}{=} (\partial_x f, \partial_y f, \partial_z f)$$

$$\nabla \cdot \vec{F} = (\partial_x, \partial_y, \partial_z) \cdot \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

$$\overset{\text{def}}{=} \partial_x F_1 + \partial_y F_2 + \partial_z F_3$$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\overset{\text{def}}{=} (\partial_y F_3 - \partial_z F_2, \partial_z F_1 - \partial_x F_3, \partial_x F_2 - \partial_y F_1)$$

Goal: $\displaystyle\int_A^B \nabla f(\vec{r})\cdot d\vec{r} = f(B) - f(A)$

$\quad\quad\quad (\nabla f \cdot \vec{T}\, ds)$

$\vec{r} = (x, y, z)$. $d\vec{r} = \vec{T}\, ds$

$A \in \mathbb{R}^3$, $B \in \mathbb{R}^3$

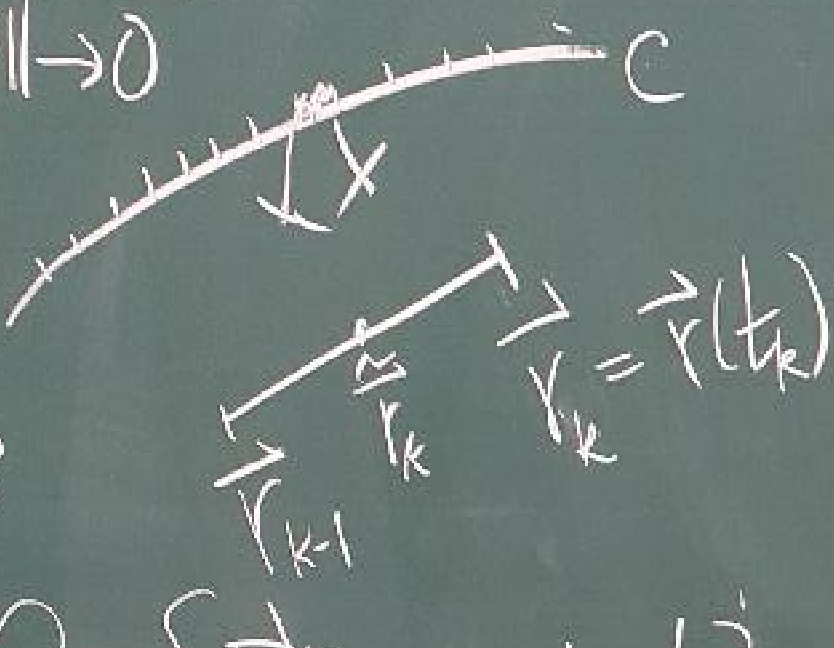
$\left(\text{Analogue of } \displaystyle\int_a^b f_x(t, y, z)\, dt = f(b, y, z) - f(a, y, z)\right)$

We first introduce related integrals.

(1) $\displaystyle\int_C f(x, y, z)\, ds = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(\tilde{\vec{r}}_k)\, \Delta S_k$

$f : D \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$

$C \subseteq D$ is a smooth curve



$\Delta S_k = |\vec{r}_k - \vec{r}_{k-1}|$, If $C = \{\vec{r}(t), a \le t \le b\}$

$\Rightarrow \Delta S_k = \dfrac{|\vec{r}_k - \vec{r}_{k-1}|}{\Delta t_k} \Delta t_k \xrightarrow{\|P\| \to 0} ds = \left|\dfrac{d\vec{r}(t)}{dt}\right| dt$

Eg $f(x,y,z) = x - 3y^2 + z$

C: line segment joining $(0,0,0)$, $(1,1,1)$

$$\int_C f(x,y,z)\,ds = ?$$

Sol: $\vec{r}(t) = (t,t,t),\ 0 \le t \le 1$

$$f(\vec{r}(t)) = t - 3t^2 + t$$

$$|\vec{r}\,'(t)| = |(1,1,1)| = \sqrt{3}$$

$$\text{Ans} = \int_0^1 (t - 3t^2 + t)\sqrt{3}\,dt = 0$$

<u>Rm</u> If we take $\vec{\tilde{r}}(\tau)=(1-\tau^2, 1-\tau^2, 1-\tau^2)$

$0\le\tau\le 1$

let $t(\tau)=1-\tau^2 \Rightarrow \hat{\vec{r}}(\tau)=\vec{r}(t(\tau))$

$$\frac{d\vec{\tilde{r}}}{d\tau}=(-2\tau,-2\tau,-2\tau)=\frac{d\vec{r}}{dt}\frac{dt}{d\tau}$$

<u>Ans</u> $= \int_{\tau=0}^{1} f(\vec{\tilde{r}}(\tau))\left|\frac{d\vec{\tilde{r}}}{d\tau}\right| d\tau$

$$=\int_{t=1}^{0} f(\vec{r}(t))\left|\frac{d\vec{r}}{dt}\right|\cdot\left|\frac{dt}{d\tau}\right| d\tau$$

$$\left|\frac{dt}{d\tau}\right|=|-2\tau|=-\frac{dt}{d\tau}$$

$$=-\int_{t=1}^{0} f(\vec{r}(t))\left|\frac{d\vec{r}}{dt}\right|\frac{dt}{d\tau}d\tau$$

$$=\int_0^1 f(\vec{r}(t))\left|\frac{d\vec{r}}{dt}\right| dt$$

Eg: $f(x,y,z) = x - 3y^2 + z$

$C = C_1 \cup C_2$

C_1: line segment joining $(0,0,0)$, $(1,1,0)$

C_2: line seg. joining $(1,1,0)$ and $(1,1,1)$

$\int_C f(x,y,z)\,ds = ?$

Sol: $C_1: \vec{r}_1(t) = (t,t,0),\ 0 \le t \le 1$

$C_2: \vec{r}_2(t) = (1,1,t),\ 0 \le t \le 1$

$$\int_C = \int_{C_1} + \int_{C_2}$$

$$\int_{C_1} f\,ds = \int_0^1 (t - 3t^2)\sqrt{2}\,dt = \frac{-\sqrt{2}}{2}$$

$$\int_{C_2} f\,ds = \int_0^1 (-2+t)\,dt = \frac{-3}{2}$$

Related integral (2)

$$\int_C \vec{F}\cdot\vec{T}\,ds$$

Also denoted as $\int_C \vec{F}(\vec{r})\cdot d\vec{r} \left(= \int_a^b \vec{F}(\vec{r}(t))\cdot \frac{d\vec{r}(t)}{dt}\,dt \right)$

$\vec{F}: D \longmapsto \mathbb{R}^3$

$(x, y, z) \longrightarrow (F_1, F_2, F_3)$

$C \subseteq D$: a smooth curve with prescribed orientation

$\vec{r}(t)$: parametrization of C

with "direction of increasing t"

= "orientation of C"

$\vec{T} = \frac{d\vec{r}(t)}{dt} \Big/ \left|\frac{d\vec{r}(t)}{dt}\right|$ unit tangent vector in the prescribed direction

Remark

$$ds = \left|\frac{d\vec{r}(t)}{dt}\right| dt$$

$$\vec{T}\,ds = \frac{d\vec{r}(t)}{dt} \Big/ \left|\frac{d\vec{r}(t)}{dt}\right| \cdot \left|\frac{d\vec{r}(t)}{dt}\right| dt$$

$$= \frac{d\vec{r}(t)}{dt}\, dt$$

$$\therefore \int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}(t)}{dt}\, dt$$

$$\left(= \int_C \vec{F}(\vec{r}) \cdot d\vec{r} \right)$$