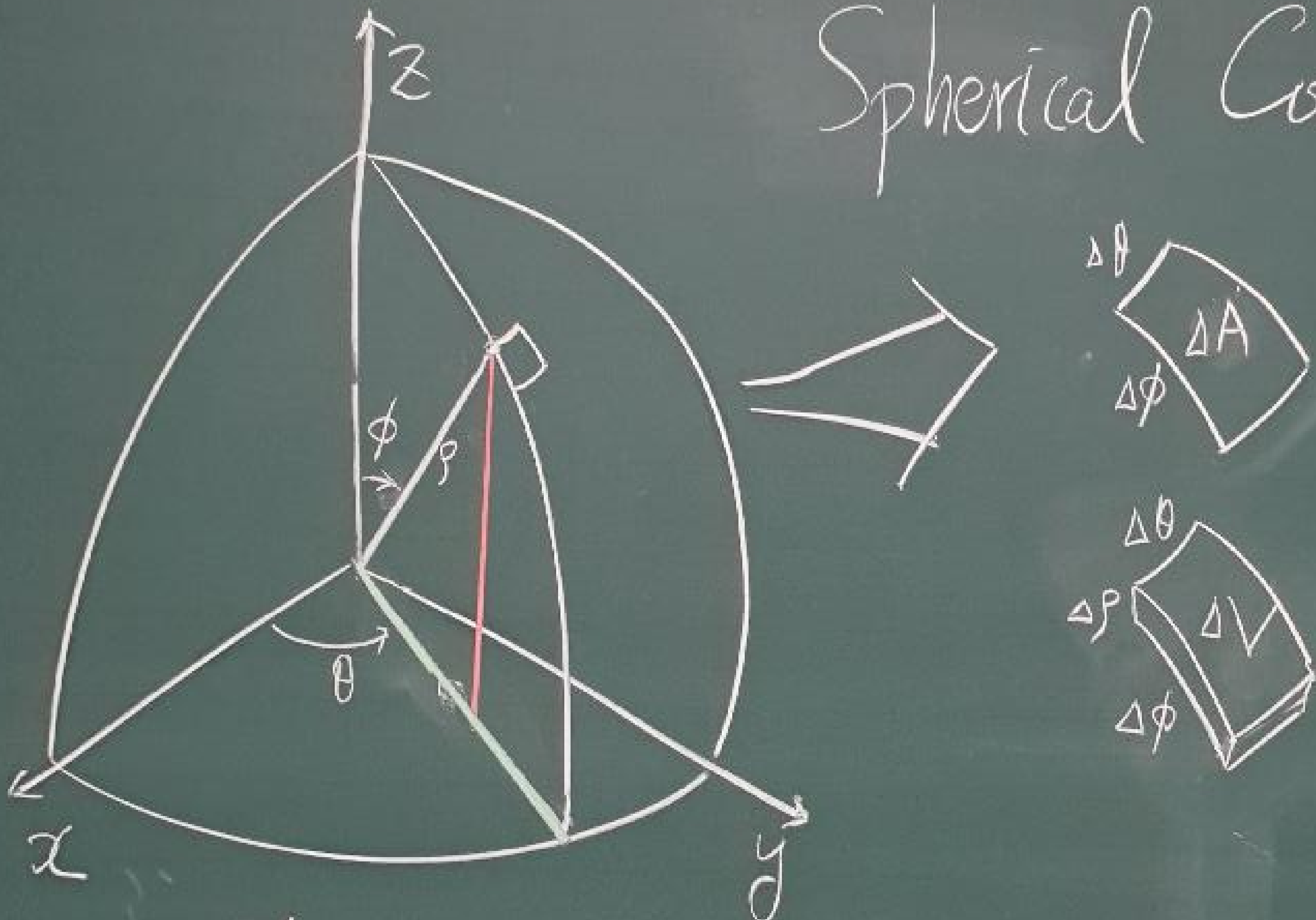
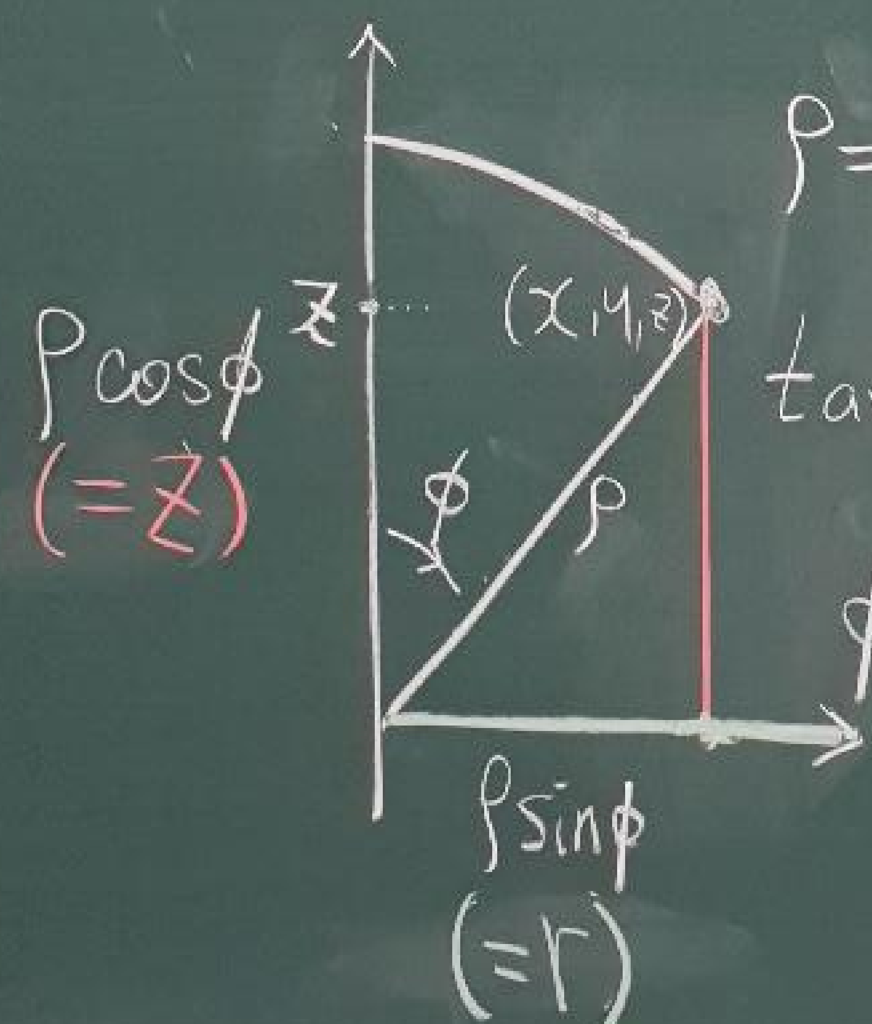
Spherical Coord.

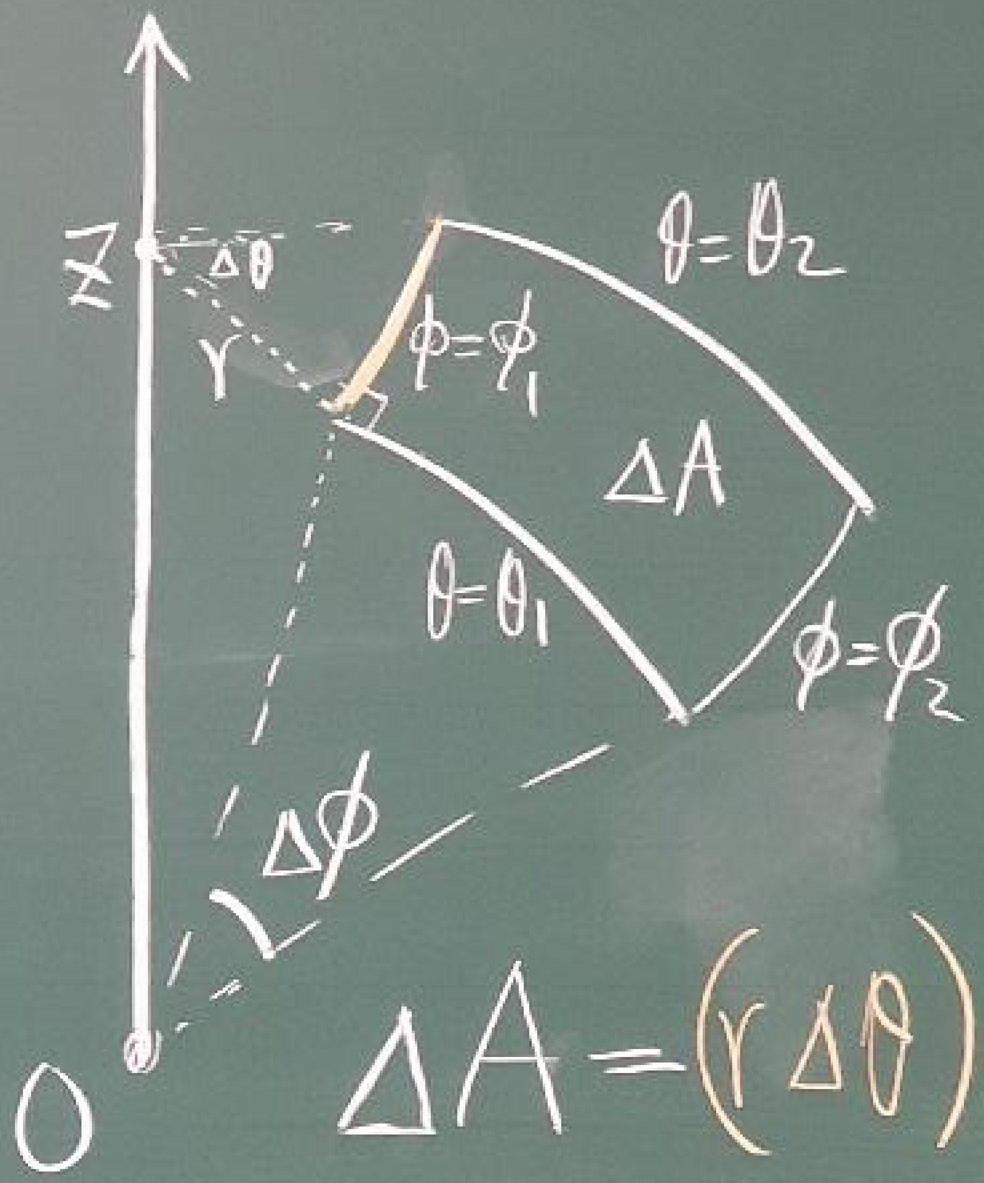




$$\rho = \sqrt{x^2+y^2+z^2}, \quad x = \rho \sin\phi \cos\theta$$

$$\tan\theta = \frac{y}{x} \qquad y = \rho \sin\phi \sin\theta$$

$$\phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \qquad z = \rho\cos\phi$$

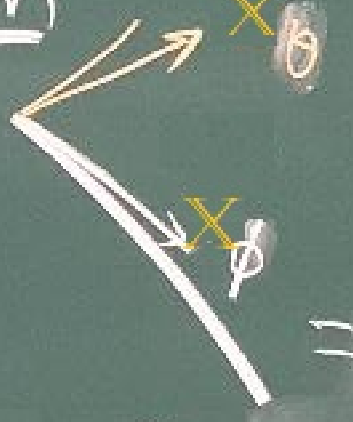


$$\Delta A = (r\,\Delta\theta)(\rho\,\Delta\phi)$$

$$= \rho \sin\phi\,\Delta\theta\;\rho\,\Delta\phi$$

$$\Delta V = \Delta A\,\Delta\rho = \underline{\underline{\rho^2 \sin\phi}}\,\Delta\rho\,\Delta\phi\,\Delta\theta$$

Rm



$$X_\phi = \frac{\partial}{\partial\phi}(x,y,z)$$

$$= (\rho C_\phi C_\theta,\ \rho C_\phi S_\theta,\ \rho S_\phi)$$

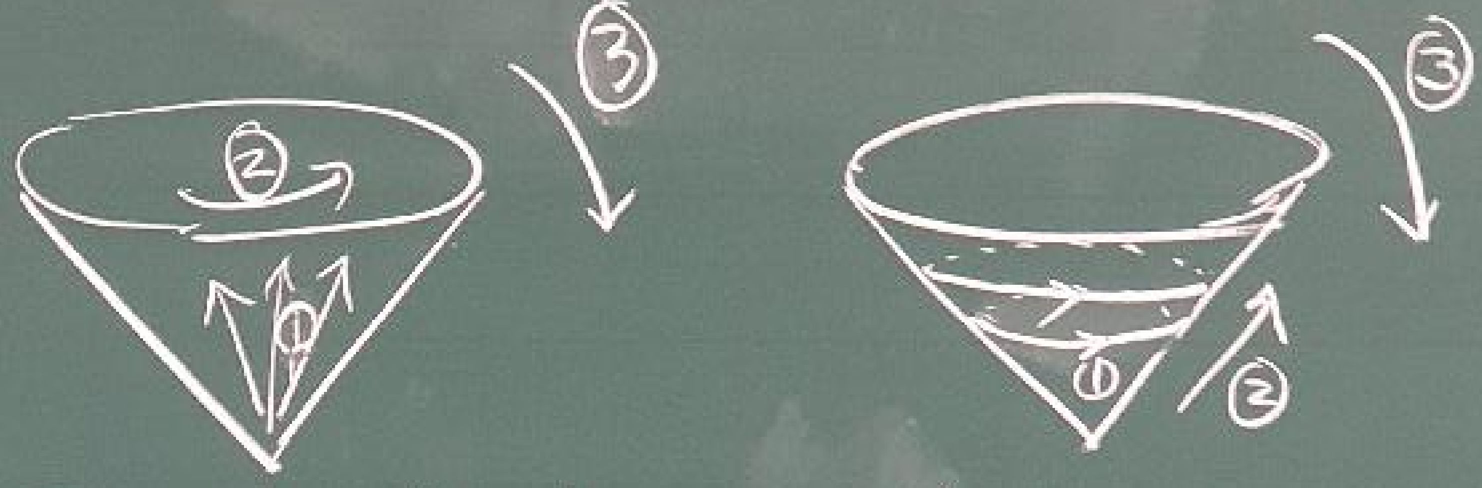
$$X_\theta = \frac{\partial}{\partial\theta}(x,y,z) = (-\rho S_\phi S_\theta,\ \rho S_\phi C_\theta,\ 0)$$

$$X_\rho = \frac{\partial}{\partial\rho}(x,y,z) = (S_\phi C_\theta,\ S_\phi S_\theta,\ C_\phi)$$

Check: they are mutually orthogonal

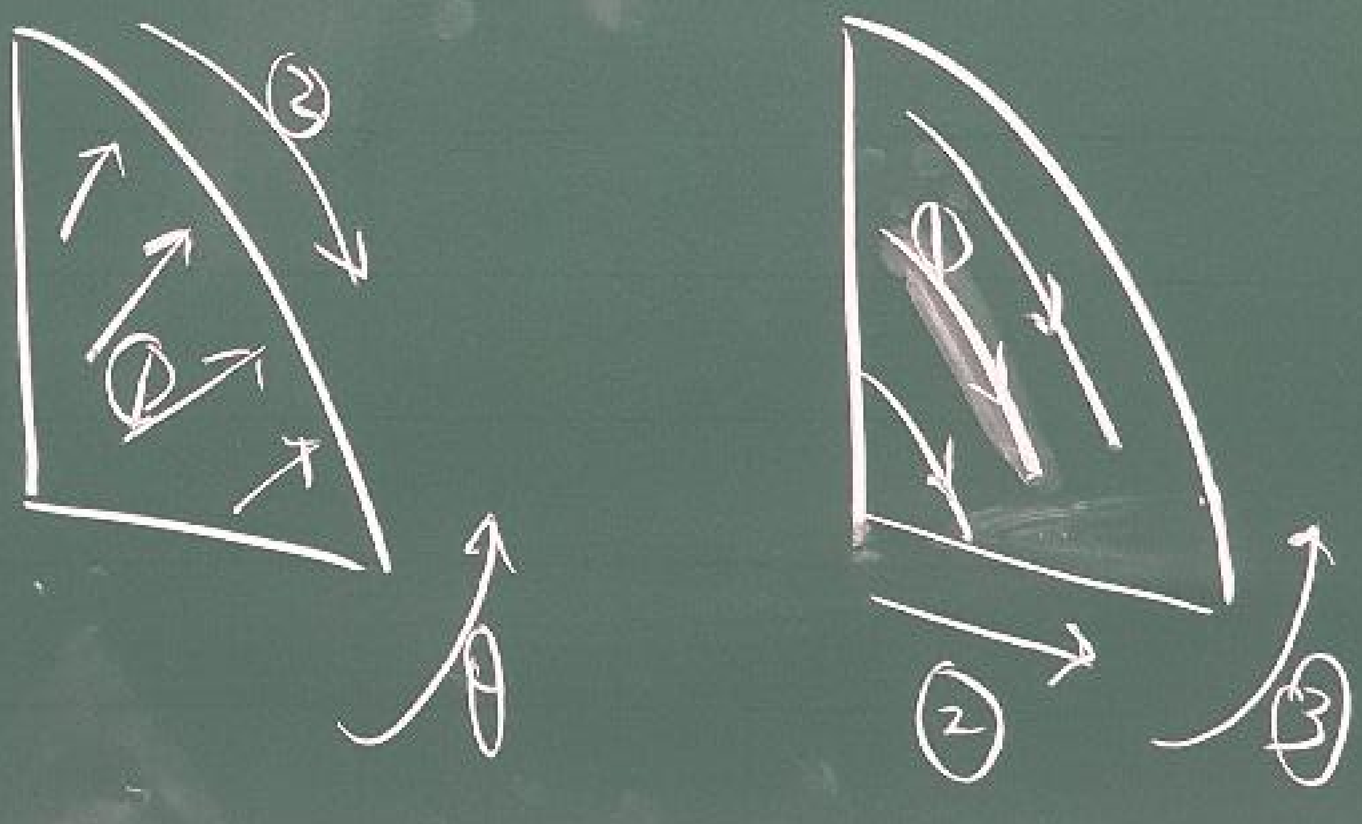
Cross sections

I: $d\rho\, d\theta\, d\phi$ or $d\theta\, d\rho\, d\phi$

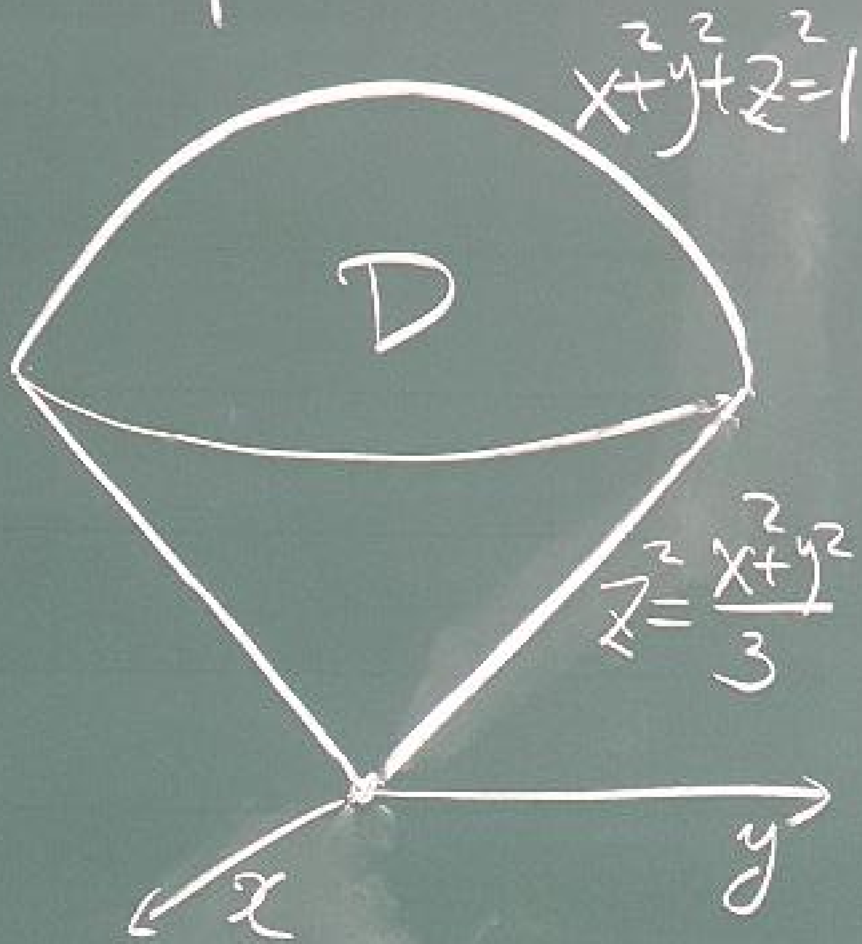


II: $d\theta\, d\phi\, d\rho$ or $d\phi\, d\theta\, d\rho$

III $d\rho\, d\phi\, d\theta$ or $d\phi\, d\rho\, d\theta$

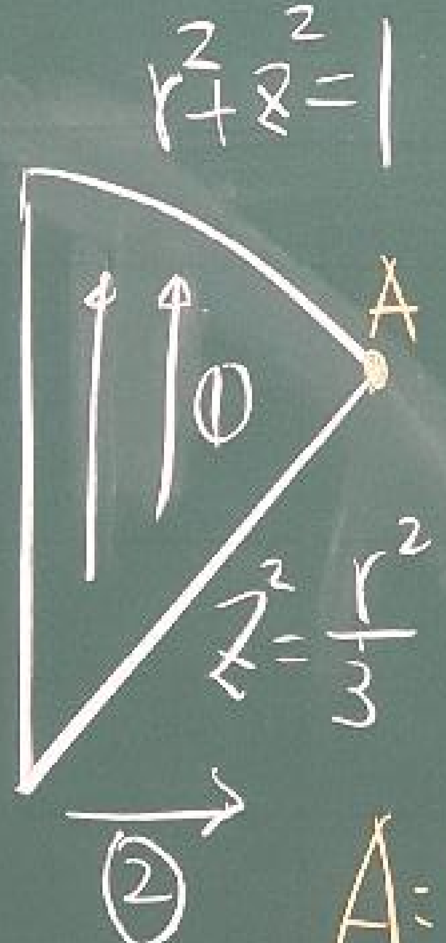


Example: Find the Volume of D:



$$\begin{cases} x^2+y^2+z^2\le 1 \\ z\ge\sqrt{\dfrac{x^2+y^2}{3}}\end{cases}$$

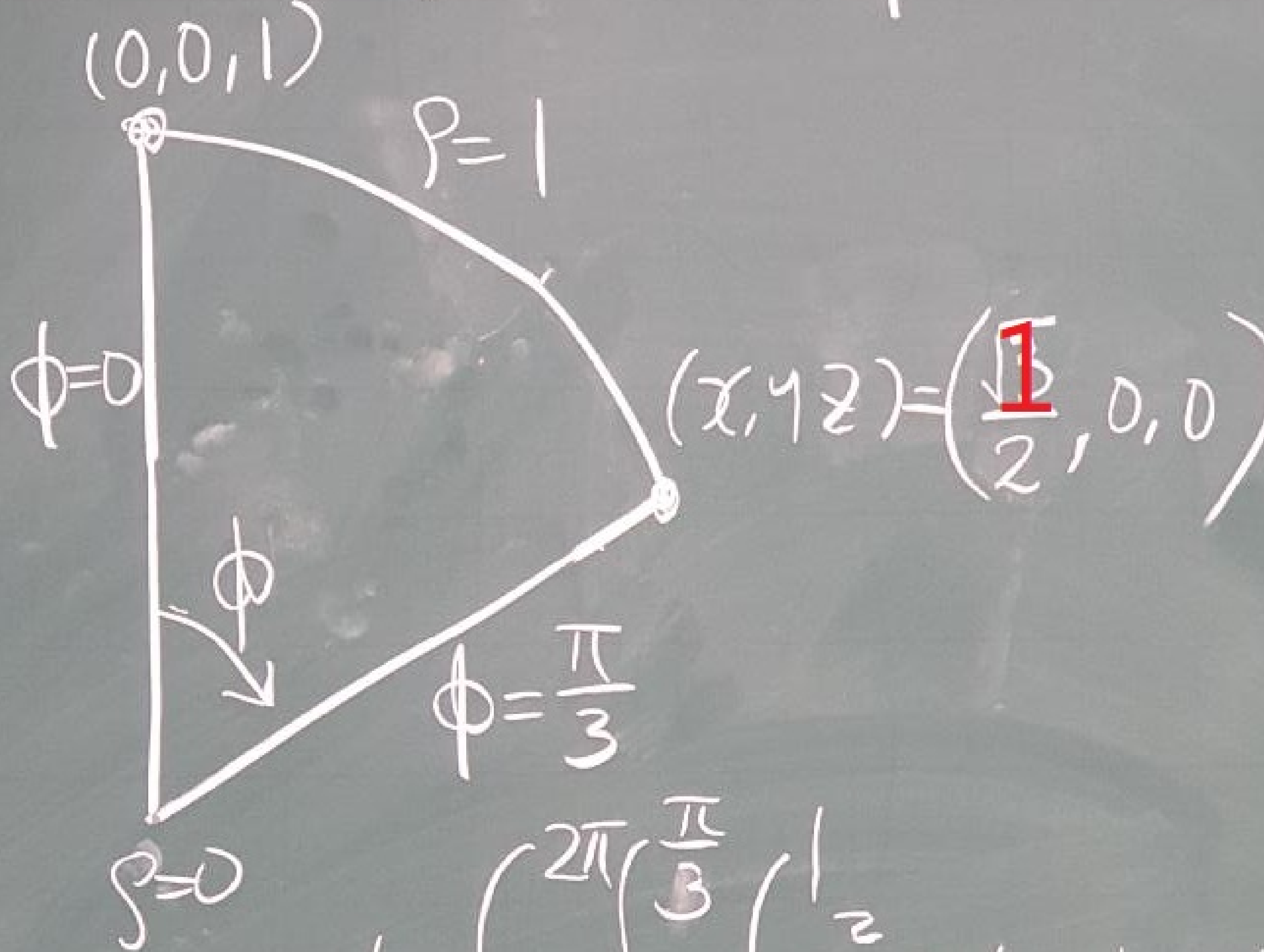
Sol (I): $dz\,dr\,d\theta$ in Cylind. coord



$$\int_0^{2\pi}\int_{r=0}^{\frac{\sqrt{3}}{2}}\int_{z=\frac{r}{\sqrt{3}}}^{\sqrt{1-r^2}} dz\; r\,dr\,d\theta$$

A: $r=\frac{\sqrt{3}}{2}$, $z=\frac{1}{2}$

(II): $d\rho\, d\phi\, d\theta$ in Spherical Coord.



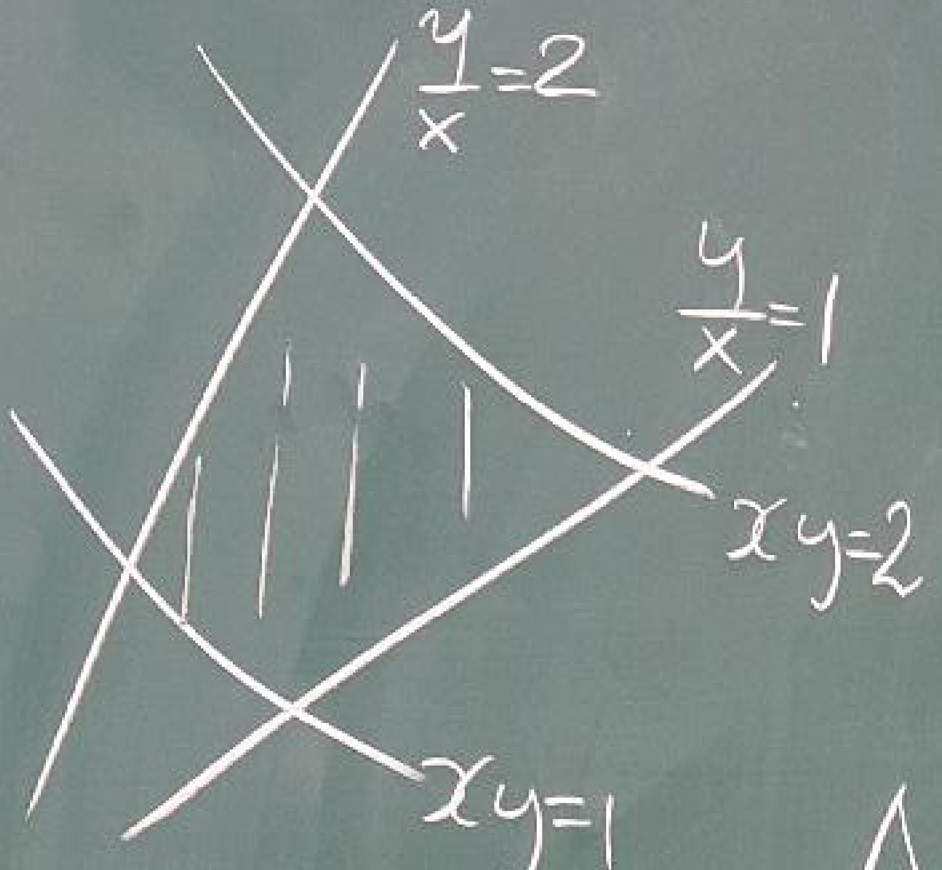
$$V=\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\frac{\pi}{3}}\int_{\rho=0}^{1}\rho^2\sin\phi\; d\rho\, d\phi\, d\theta$$

$$=\frac{1}{3}\cdot\left(1-\cos\frac{\pi}{3}\right)(2\pi)$$

$$=\frac{\pi}{3}$$

Substitution in Multiple Integrals

Eg: Find area of $R = \left\{ \begin{matrix} 1 \le xy \le 2 \\ 1 \le \frac{y}{x} \le 2 \end{matrix} \right\}$ (in I)



Sol: Let $u = xy$

$v = \frac{y}{x}$

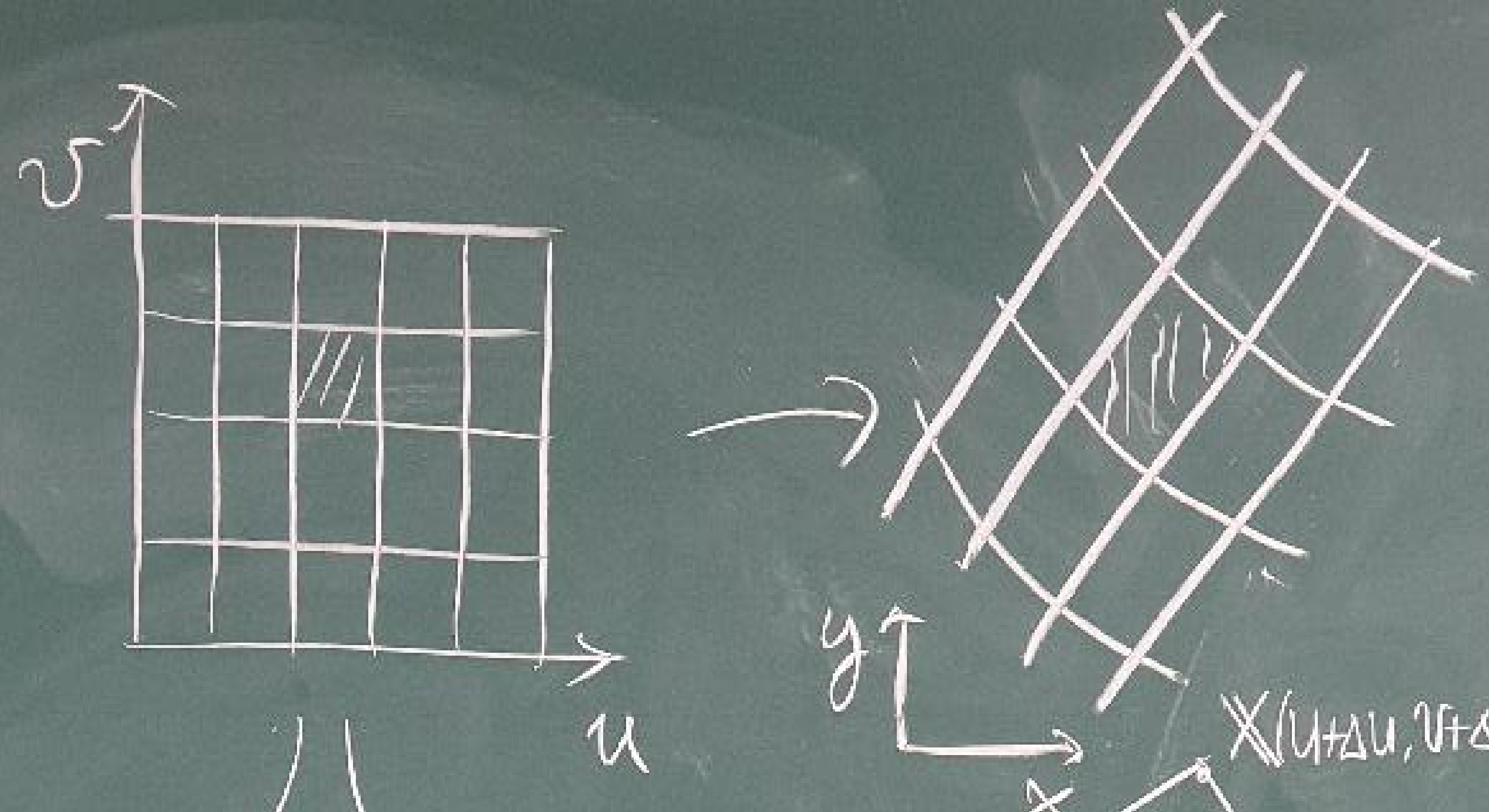
$$A = \int_{v=1}^{2} \int_{u=1}^{2} dA$$

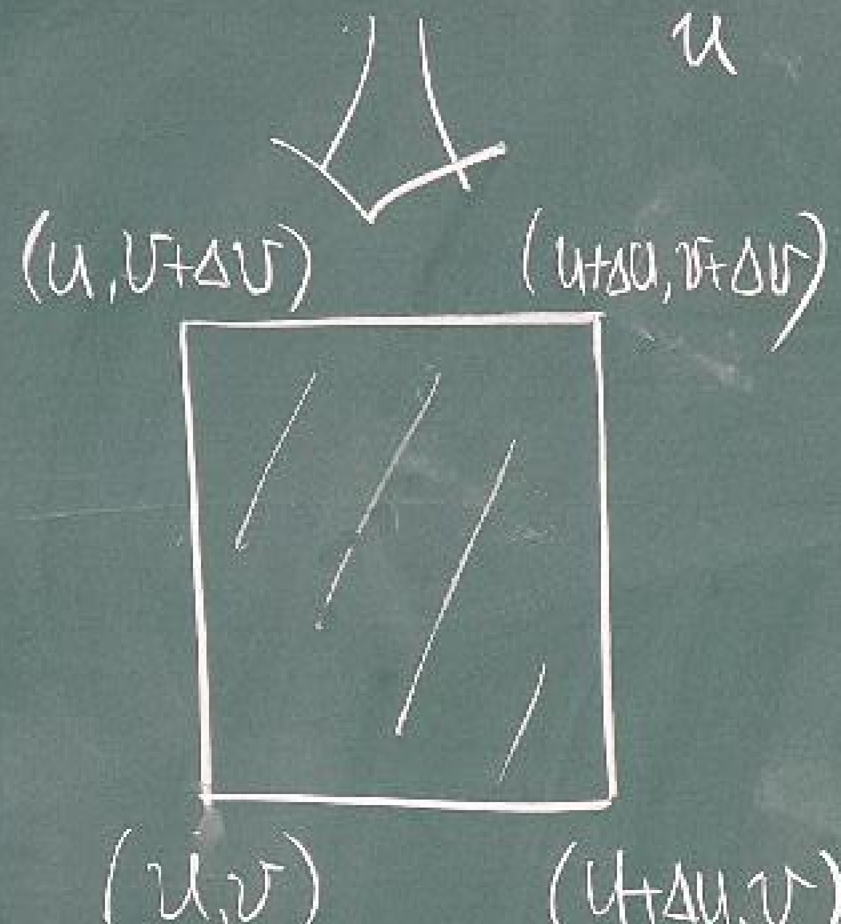
$$dA = \,?\, du\,dv$$

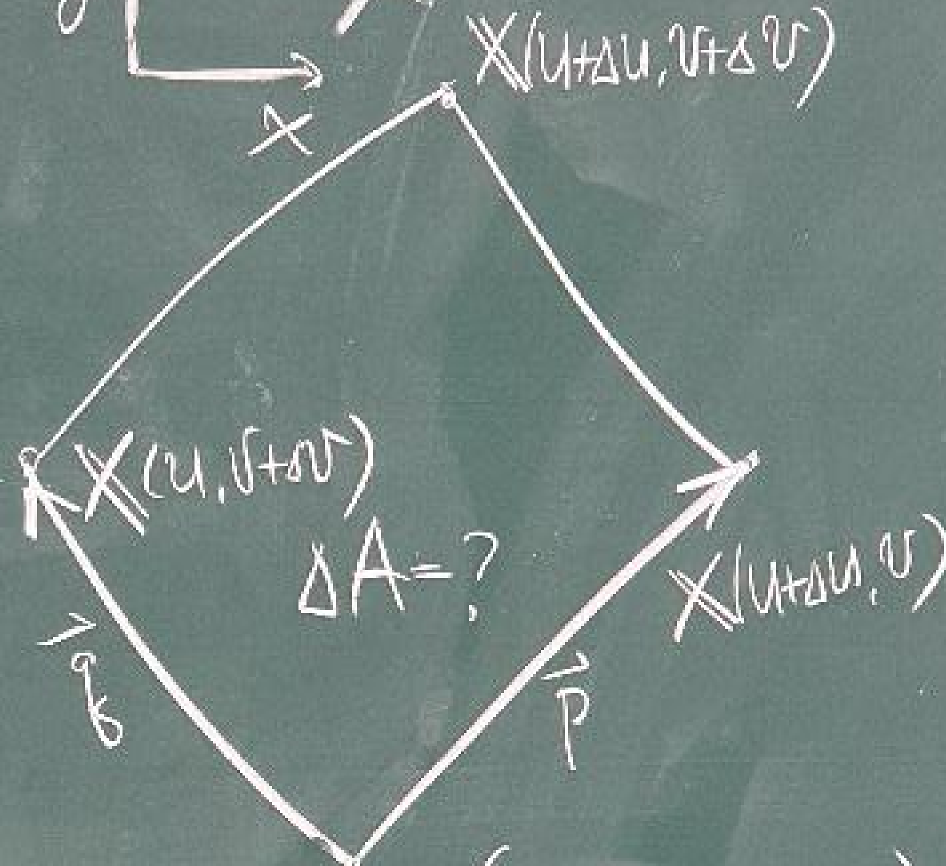
$$\begin{matrix} u = xy \\ v = y/x \end{matrix} \Rightarrow \begin{matrix} x = \sqrt{\frac{u}{v}} \\ y = \sqrt{uv} \end{matrix} \Rightarrow \begin{matrix} x_u = \frac{1}{2\sqrt{uv}}, \; x_v = \frac{-1}{2}\sqrt{\frac{u}{v^3}} \\ y_u = \frac{1}{2}\sqrt{\frac{v}{u}}, \; y_v = \frac{1}{2}\sqrt{\frac{u}{v}} \end{matrix}$$

$$J \overset{\text{(see next slide)}}{=} \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \frac{1}{2v}$$

$$A = \int_1^2 \int_1^2 \left| \frac{1}{2v} \right| du\,dv = \frac{\ln 2}{2}$$







$$\mathbf{X}(u,v) = (x(u,v),\, y(u,v))$$

$$\Delta A = \left|\vec{p} \times \vec{q}\right| = \left|\underbrace{\left(\mathbf{X}(u+\Delta u, v) - \mathbf{X}(u,v)\right)}_{\vec{p}} \times \underbrace{\left(\mathbf{X}(u, v+\Delta v) - \mathbf{X}(u,v)\right)}_{\vec{q}}\right|$$

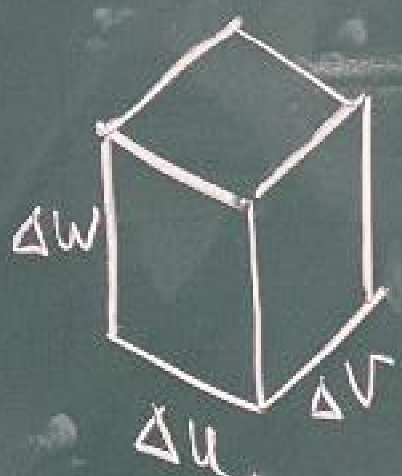
$$\vec{p} = \frac{\partial \mathbf{X}}{\partial u}(u + c_1 \Delta u, v)\,\Delta u, \quad \vec{q} = \frac{\partial \mathbf{X}}{\partial v}(u, v + c_2 \Delta v)\,\Delta v$$

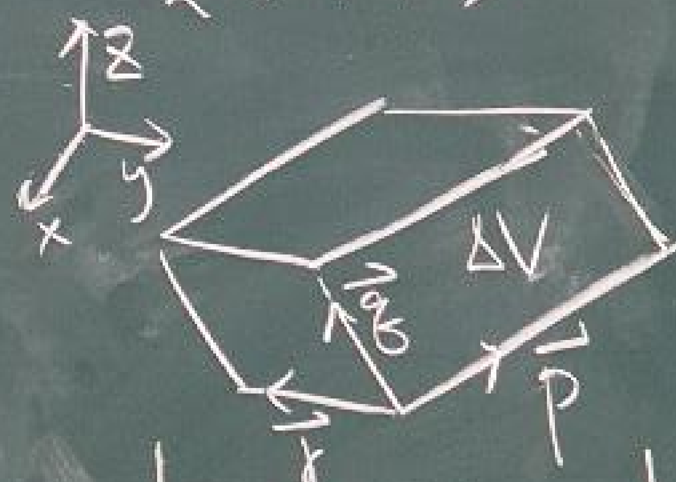
$$\Delta A = \left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right| \Delta u \Delta v = |J| = \Delta u \Delta v$$

$$J = \begin{vmatrix} * & x \\ y & x \end{vmatrix}$$

For triple integrals

$$\begin{matrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{matrix} \iff \begin{matrix} x(u,v,w) \\ y(u,v,w) \\ z(u,v,w) \end{matrix}$$





$$\Delta V = \left| \vec{p} \times \vec{q} \cdot \vec{r} \right| = \left| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix} \cdot \vec{r} \right|$$

$$= \left| \begin{vmatrix} r_1 & r_2 & r_3 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix} \right|$$

$$\vec{p} = \mathbb{X}(u+\Delta u, v, w) - \mathbb{X}(u,v,w) = \mathbb{X}_u(\cdot)\,\Delta u$$

$$\vec{q} = \mathbb{X}(u, v+\Delta v, w) - \mathbb{X}(u,v,w) = \mathbb{X}_v(\cdot)\,\Delta v$$

$$\vec{r} = \mathbb{X}(u, v, w+\Delta w) - \mathbb{X}(u,v,w) = \mathbb{X}_w(\cdot)\,\Delta w$$

$$\Delta V = |J|\,\Delta u\,\Delta v\,\Delta w, \quad J = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$dV = |J|\,du\,dv\,dw$$