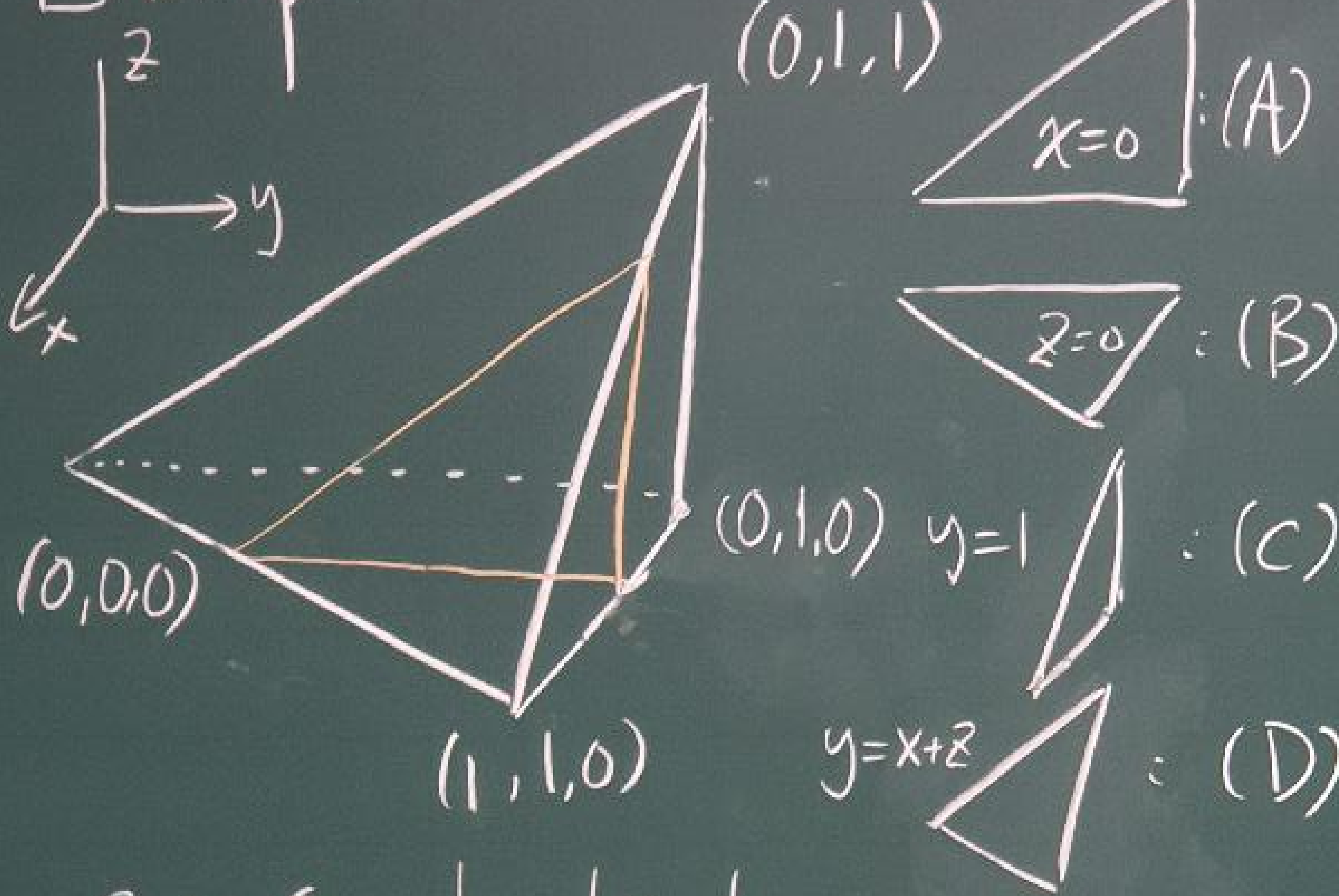
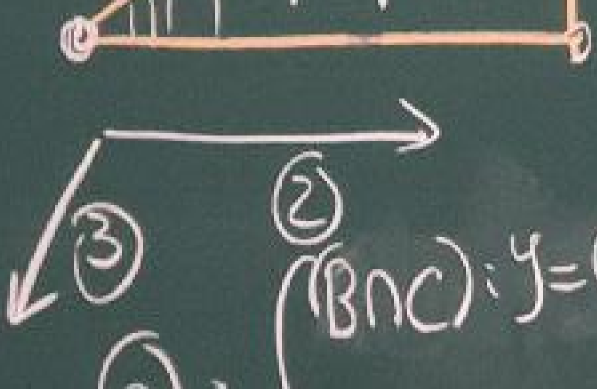
Example : Volume



Case ③ : $dz\,dy\,dx$



①: $$\int_{(B):\, z=F_1(x,y)}^{(D):\, z=F_2(x,y)} dz = \int_{z=0}^{y-x} dz$$

②: $$\int_{(B\cap D):\, y=G_1(x)}^{(B\cap C):\, y=G_2(x)} \int_{z=0}^{y-x} dz\,dy = \int_{y=x}^{y=1} \int_{z=0}^{y-x} dz\,dy$$

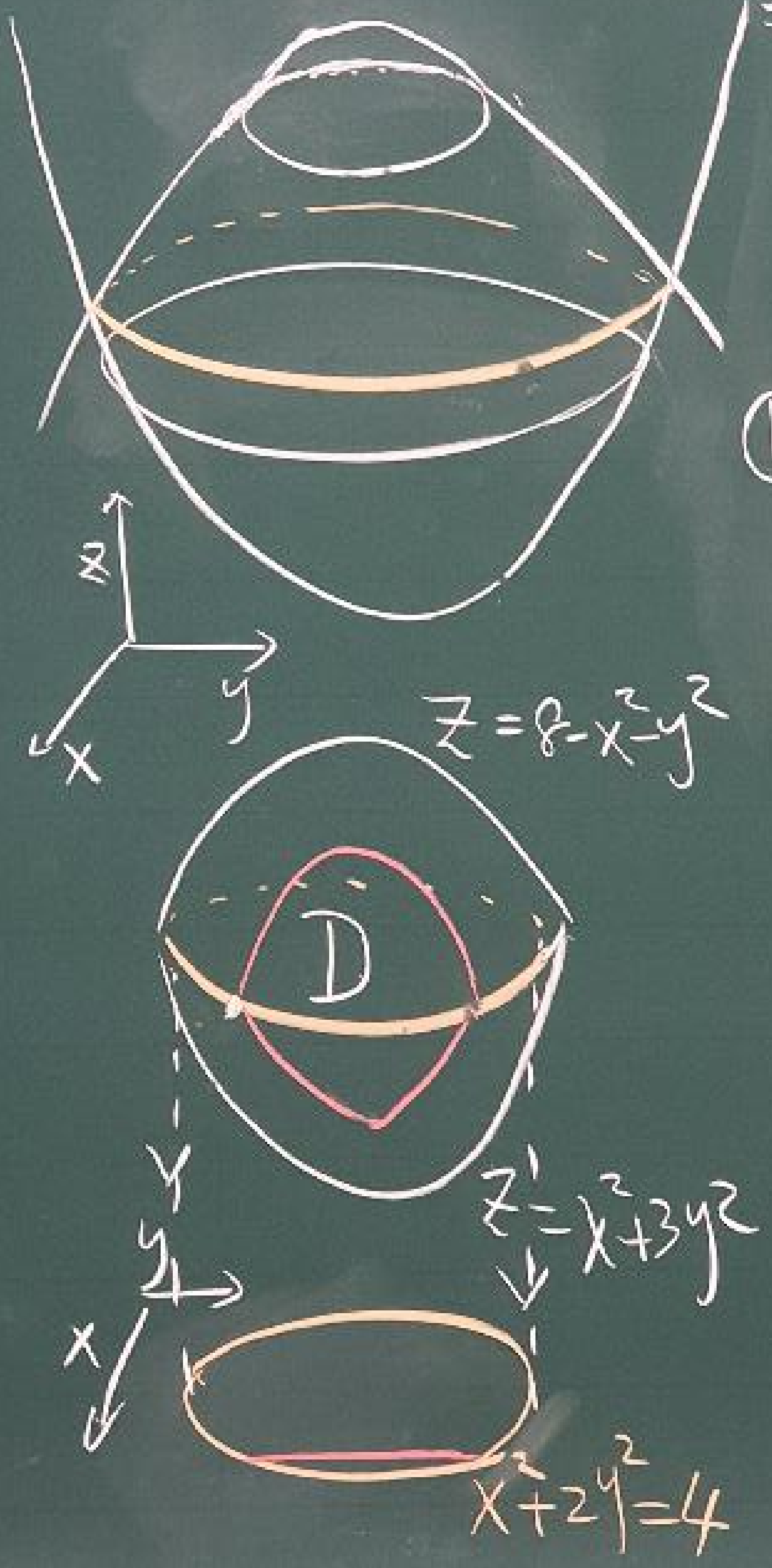
③: $$\int_{x=0}^{1}\int_{y=x}^{1}\int_{z=0}^{y-x} dz\,dy\,dx$$

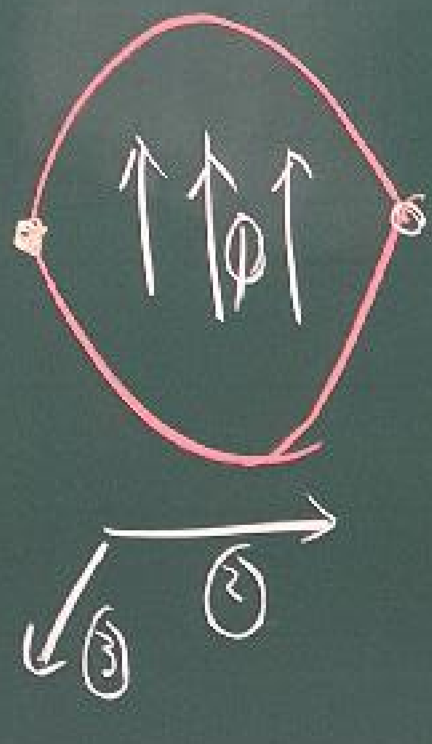
Example: Find the volume of the domain $x^2+3y^2 \le z \le 8-x^2-y^2$

Sol: For example

$dz\,dy\,dx$

①: $$\int_{z=x^2+3y^2}^{z=8-x^2-y^2} dz$$





② $\int_{y=G_1(x)}^{G_2(x)} \int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy$

$G_1(x) = ?\quad G_2(x) = ?$

$G_1(x); G_2(x): \begin{cases} x^2+3y^2 = z \\ 8-x^2-y^2 = z \end{cases}$

Eliminate z

$\Rightarrow x^2+2y^2 = 4$

$\therefore y = \pm\sqrt{\frac{4-x^2}{2}}$

$② = \int_{y=-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{z=x^2+3y^2}^{8-x^2-y^2} dz\,dy$

③ $x_{min}, x_{max} : \{x^2+2y^2 \leq 4\}$
$\quad\;\; \overset{\shortparallel}{-2} \quad\;\; \overset{\shortparallel}{2}$

$\int_{-2}^{2} \int_{y=-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{z=x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$

Triple integrals in Cylindrical and Spherical Coordinates

1. Cylindrical coordinates

= polar coordinate + z coord.

$$(x, y, z) \longleftrightarrow (r, \theta, z)$$

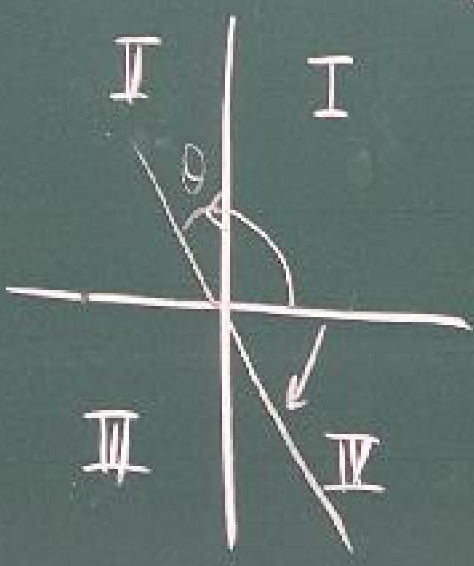
$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

$$r = \sqrt{x^2+y^2}, \quad \tan\theta = \frac{y}{x}, \quad z = z$$

$$\theta = \tan^{-1}\frac{y}{x} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right), \quad (x,y) \in \frac{\mathrm{I}}{\mathrm{IV}}$$

$$\theta = \tan^{-1}\frac{y}{x} + \pi \in \left(\frac{\pi}{2}, \pi\right) \quad (x,y) \in \mathrm{II}$$

$$\left(\pi, \frac{3\pi}{2}\right) \qquad \mathrm{III}$$



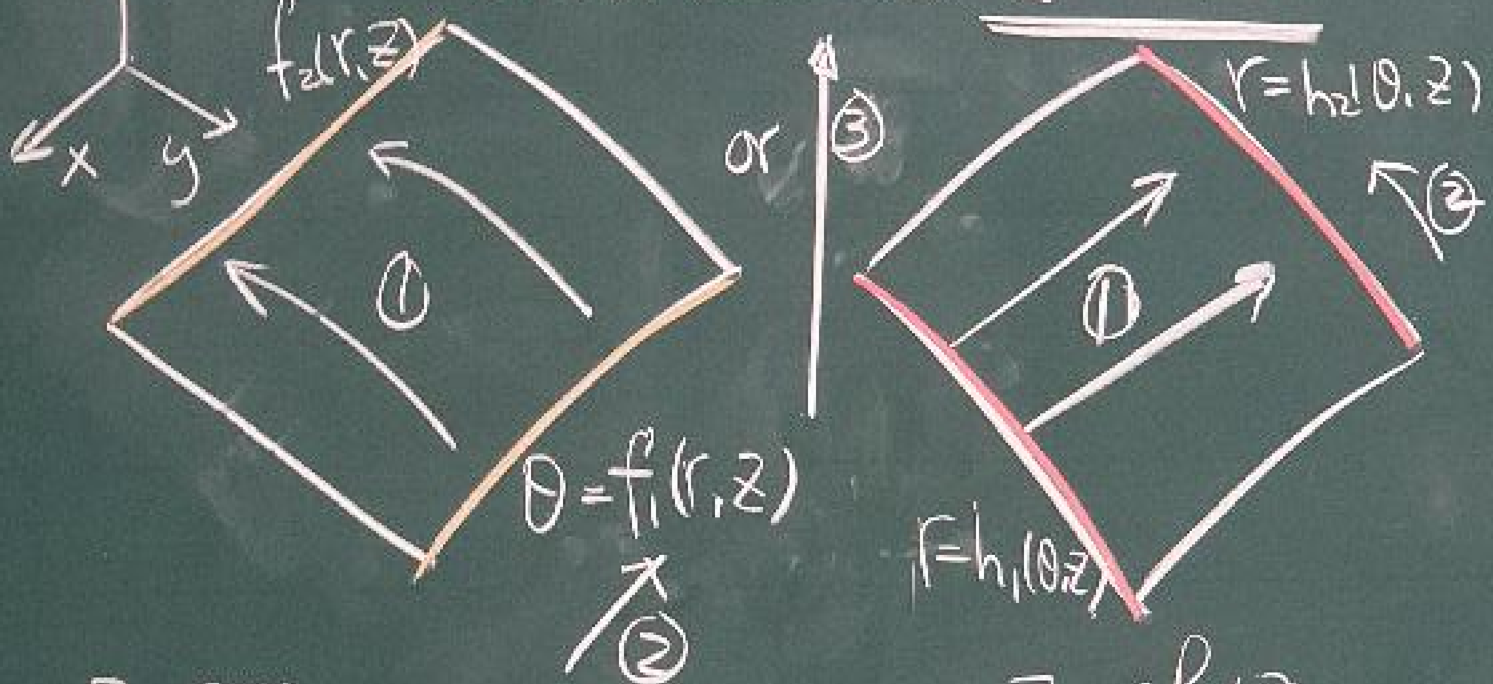
$$dV = dx\,dy\,dz$$

$$= r\,dr\,d\theta\,dz$$

How to determine limits of integration?

Case I: $d\theta\, r\,dr\,dz$ or $r\,dr\,d\theta\,dz$

Cross section: $z = \text{const}$



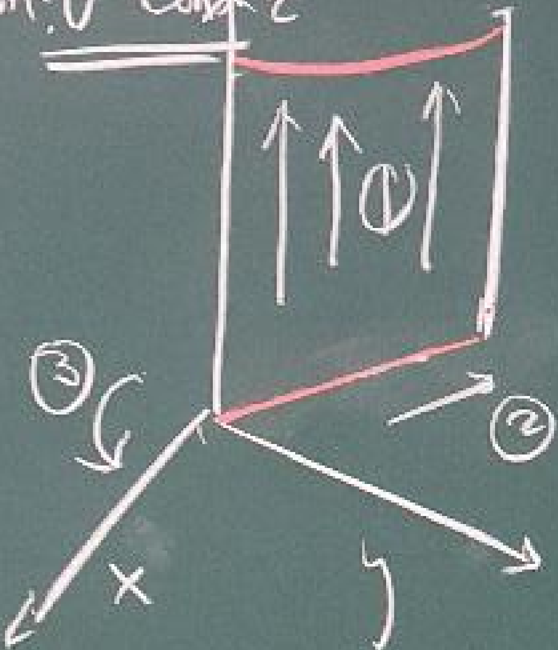
$$\int_{z=z_1}^{z_2}\int_{r=g_1(z)}^{g_2(z)}\int_{\theta=f_1(r,z)}^{f_2(r,z)} (*)\, d\theta\, r\,dr\,dz$$

or

$$\int_{z=z_1}^{z_2}\int_{\theta=k_1(z)}^{k_2(z)}\int_{r=h_1(\theta,z)}^{h_2(\theta,z)} (*)\, r\,dr\,d\theta\,dz$$

Case II $r\,dr\,dz\,d\theta$ or $dz\,r\,dr\,d\theta$

cross section: $\underline{\theta = \text{const}}$

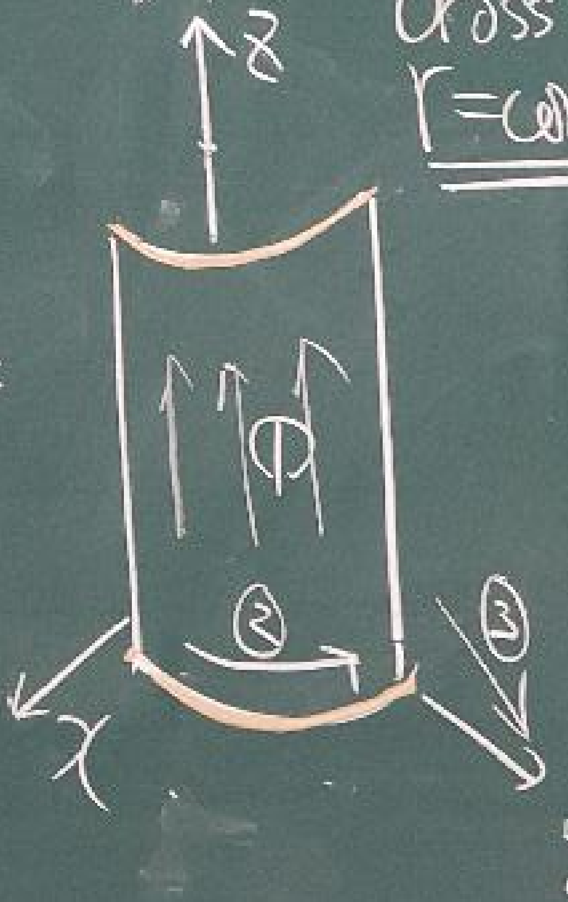


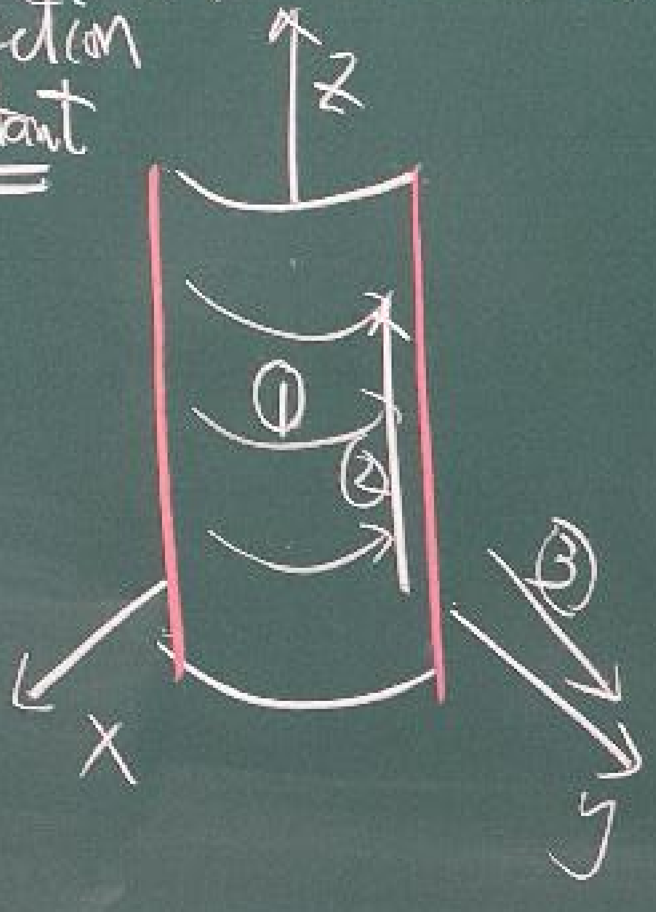
$$\int_{\textcircled{3}_1}^{\textcircled{3}_2} \int_{\textcircled{2}=G_1(\textcircled{3})}^{G_2(\textcircled{3})} \int_{\textcircled{1}=F_1(\textcircled{2},\textcircled{3})}^{F_2(\textcircled{2},\textcircled{3})} (*)\, d\textcircled{1}\, d\textcircled{2}\, d\textcircled{3}$$

Case III $dz\,d\theta\,r\,dr$ or $d\theta\,dz\,r\,dr$

cross section

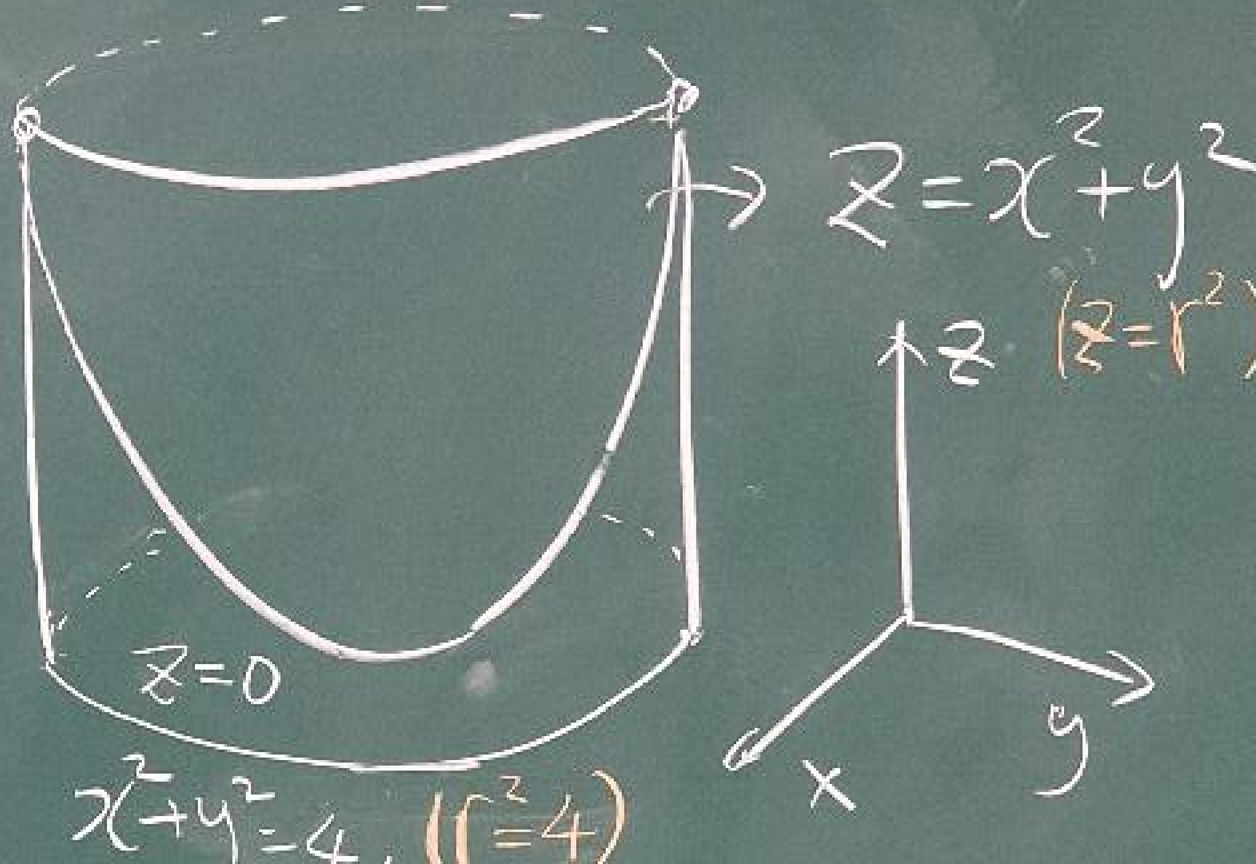
$\underline{r = \text{constant}}$





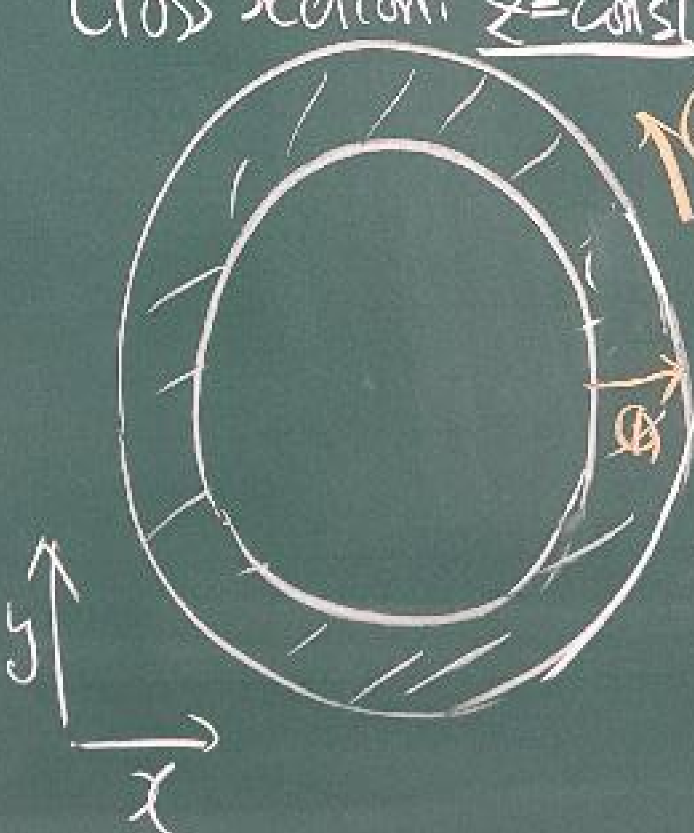
Example $D=\left\{\begin{matrix} x^2+y^2\le 4 \\ 0\le z\le x^2+y^2\end{matrix}\right\}$

Find volume of D using cyl. coord.



Case I: $r\,dr\,d\theta\,dz$

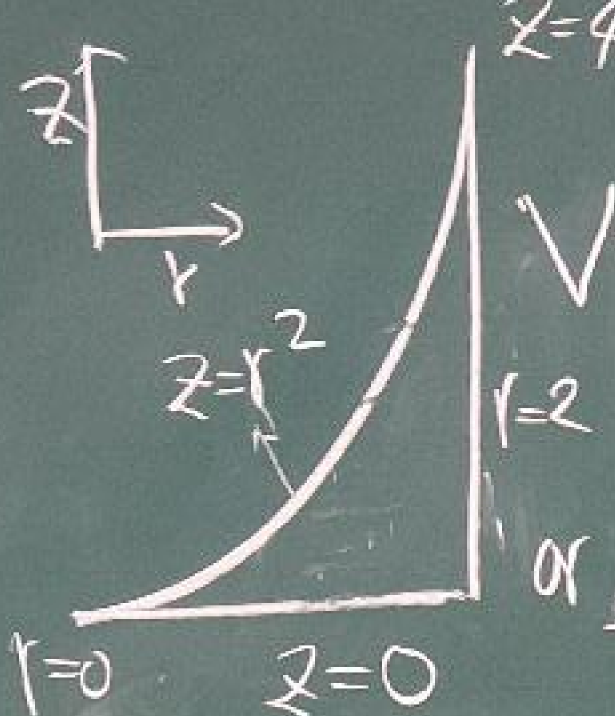
Cross section: $\underline{z=\text{const}}$



$$\int_{z=0}^{4}\int_{0}^{2\pi}\int_{r=\sqrt{z}}^{2} r\,dr\,d\theta\,dz$$

Case II $r\,dr\,dz\,d\theta$ or $dz\,r\,dr\,d\theta$

cross section: $\underline{\theta = \text{constant}}$

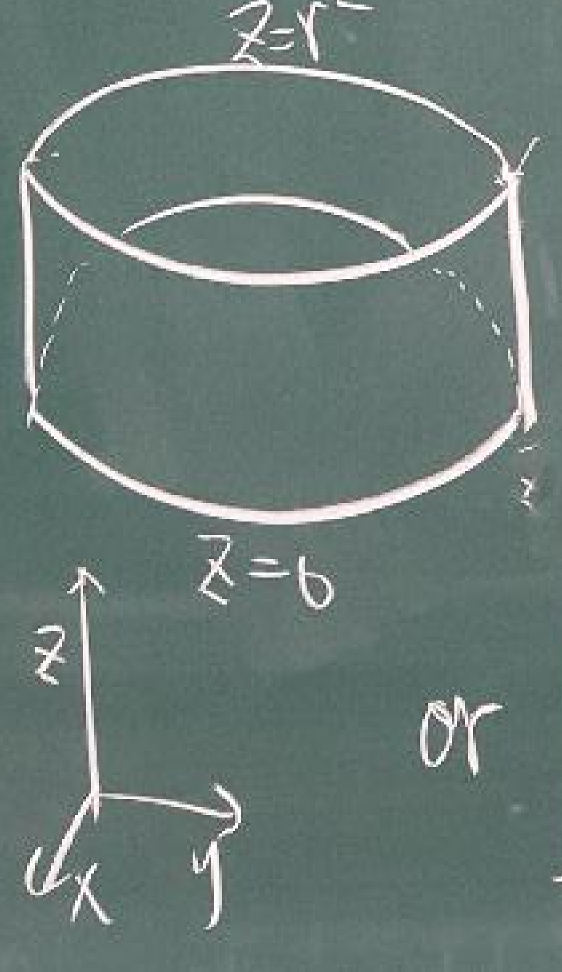


$$V=\int_{0}^{2\pi}\int_{z=0}^{4}\int_{r=\sqrt{z}}^{2} r\,dr\,dz\,d\theta$$

$$\text{or} = \int_{0}^{2\pi}\int_{r=0}^{2}\int_{z=0}^{r^2} dz\,r\,dr\,d\theta$$

Case III $dz\,d\theta\,r\,dr$ or $d\theta\,dz\,r\,dr$

cross section: $r = \text{constant}$



$$\int_{r=0}^{2}\int_{\theta=0}^{2\pi}\int_{z=0}^{r^2} dz\,d\theta\,r\,dr$$

$$\text{or} \int_{r=0}^{2}\int_{z=0}^{r^2}\int_{0}^{2\pi} d\theta\,dz\,r\,dr$$