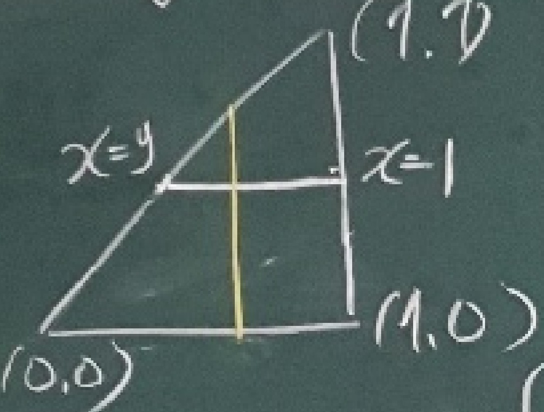
Example: R = region bounded by $x=y$, $x=1$, $y=0$



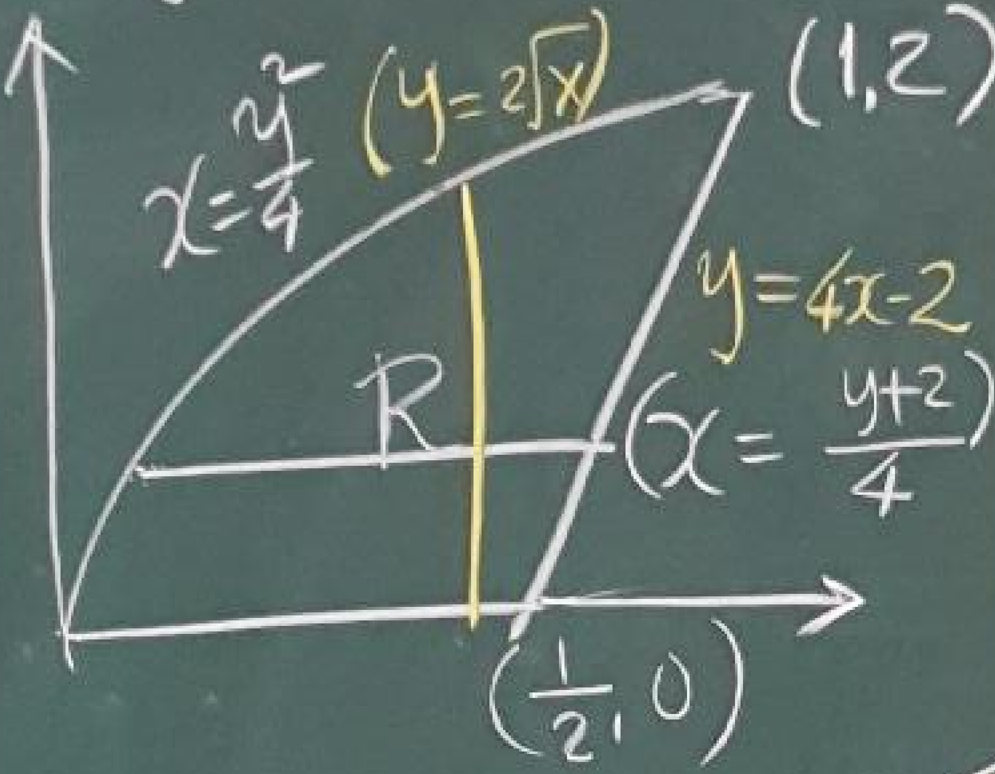
$$\iint_R \frac{\sin x}{x}\, dA = ?$$

(i) $\iint dx\, dy$: $\displaystyle\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy = ?$

(ii) $\iint dy\, dx$: $\displaystyle\int_0^1 \int_0^x \frac{\sin x}{x}\, dy\, dx$

$$= \int_0^1 (x-0)\frac{\sin x}{x}\, dx = 1 - \cos 1$$

Example Find area bounded by $y=0$, $y=4x-2$, $x=\frac{y^2}{4}$



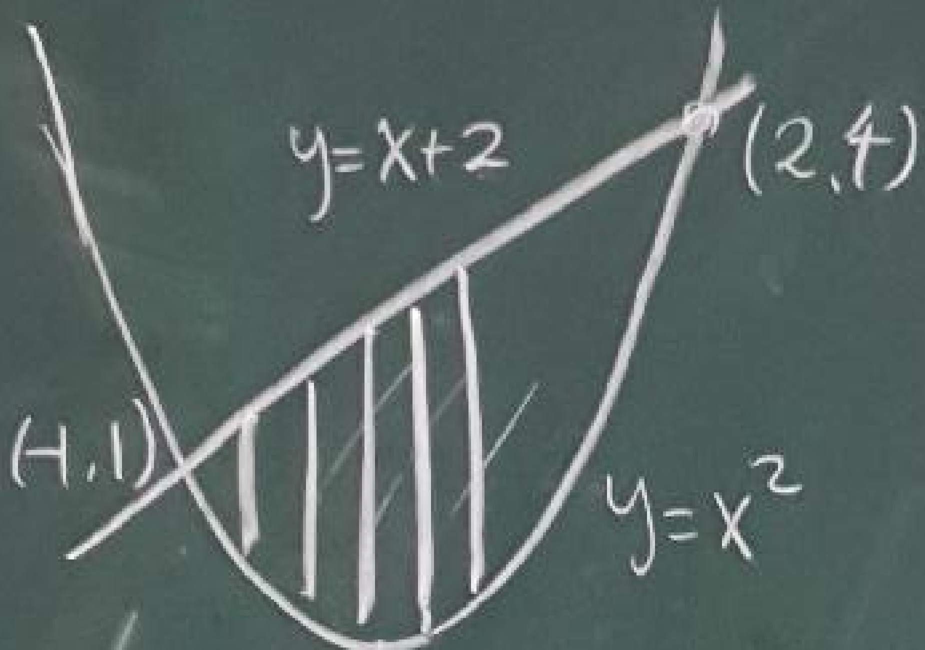
$$\boxed{\text{Area of } R = \iint_R 1\,dA}$$

 =  + 

$$= \int_0^{\frac{1}{2}}\left(\int_0^{2\sqrt{x}} 1\,dy\right)dx + \int_{\frac{1}{2}}^{1}\left(\int_{4x-2}^{2\sqrt{x}} 1\,dy\right)dx$$

or =  $\displaystyle\int_0^2\left(\int_{\frac{y^2}{4}}^{\frac{y+2}{4}} 1\,dx\right)dy$

Example Find area bounded by $y = x^2$ and $y = x + 2$

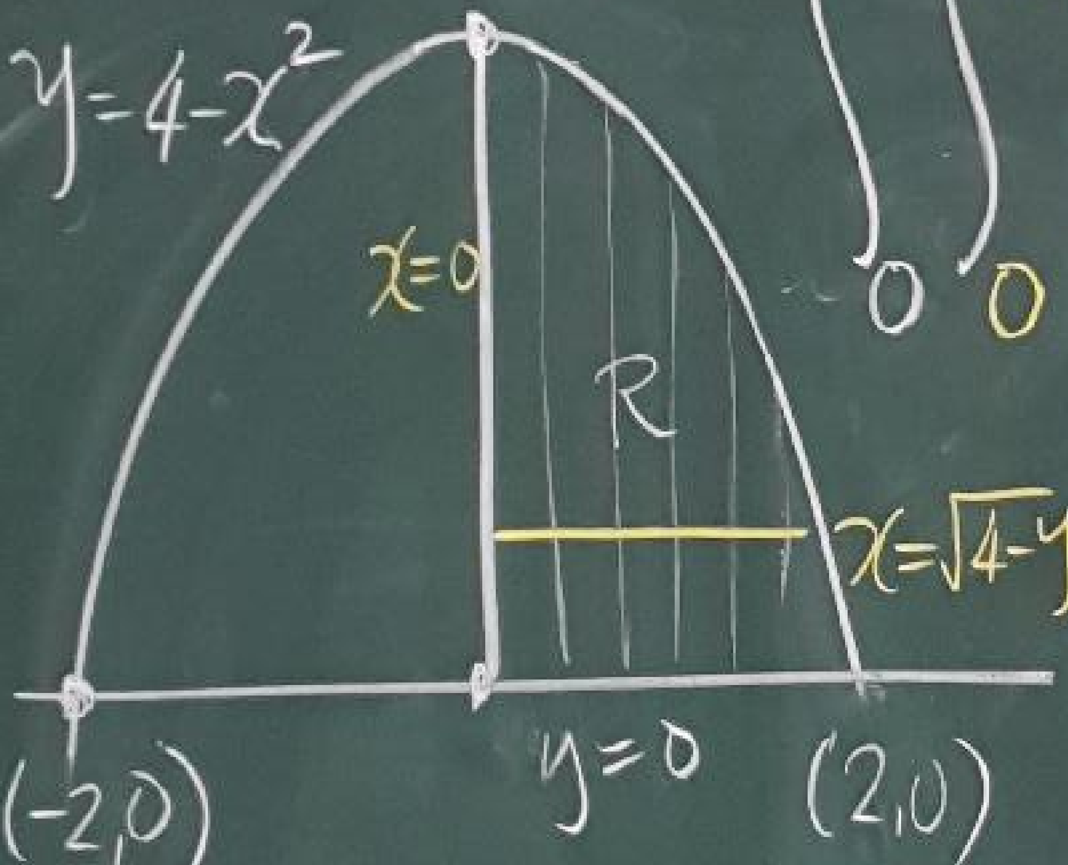


$$\int_{-1}^{2} \int_{x^2}^{x+2} 1 \, dy \, dx$$

$$= \int_{-1}^{2} (x + 2 - x^2) \, dx = \frac{9}{2}$$

Example $\displaystyle\int_0^2 \underbrace{\int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dy}_{\text{difficult}}\,dx = ?$

Sol: Try $dx\,dy$:



$$\int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\,dx\,dy$$

$$= \int_0^4 \frac{e^{2y}}{4-y} \cdot \left(\frac{x^2}{2}\right)\Bigg|_0^{\sqrt{4-y}} dy$$

$$= \frac{1}{2}\int_0^4 e^{2y}\,dy$$

$$= \frac{1}{4}e^{2y}\Big|_0^4 = \frac{e^8-1}{4}$$

Integration in polar coordinate

Eg: $\iint_R f(x,y)\,dA$, R = region bounded by $y = x$, $x^2+y^2=1$, $y=\sqrt{3}x$, $x^2+y^2=4$

Both $\int_*^* \int_?^? f(x,y)\,dx\,dy$

and $\int_*^* \int_?^? f(x,y)\,dy\,dx$

are complicated (need to break into several integrals)

In this example

$$R=\left\{(r,\theta)\ \middle|\ 1\leq r\leq 2,\ \frac{\pi}{4}\leq\theta\leq\frac{\pi}{3}\right\}$$

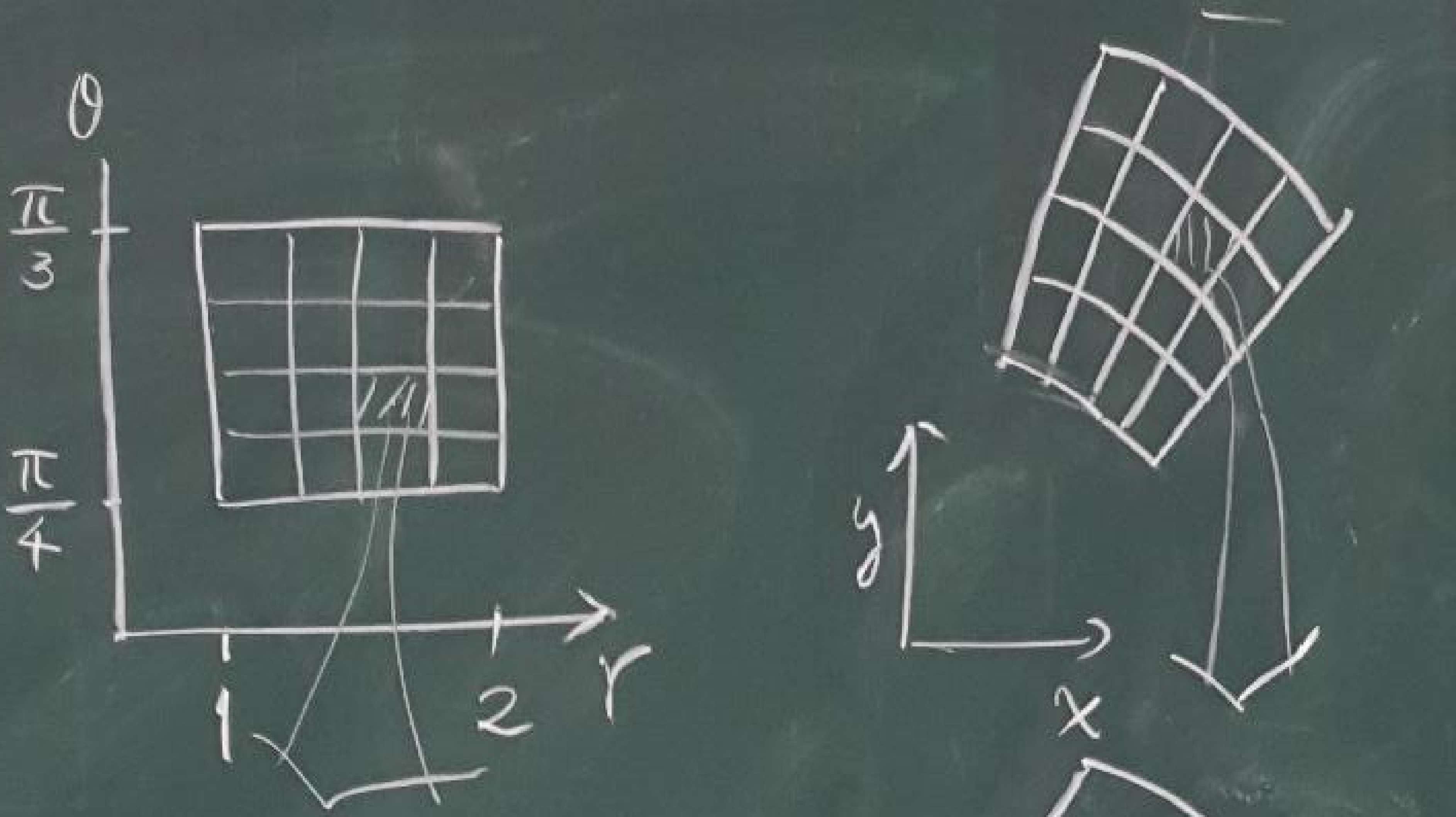
So either $dr\,d\theta$ or $d\theta\,dr$ are convenient

$$I=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\int_{r=1}^{2} f(\overset{x}{\overset{\downarrow}{r\cos\theta}},\ \overset{y}{\overset{\downarrow}{r\sin\theta}})\,dA$$

$$dA=\ ?\ dr\,d\theta$$

<u>Ans</u>: $dA=r\,dr\,d\theta$

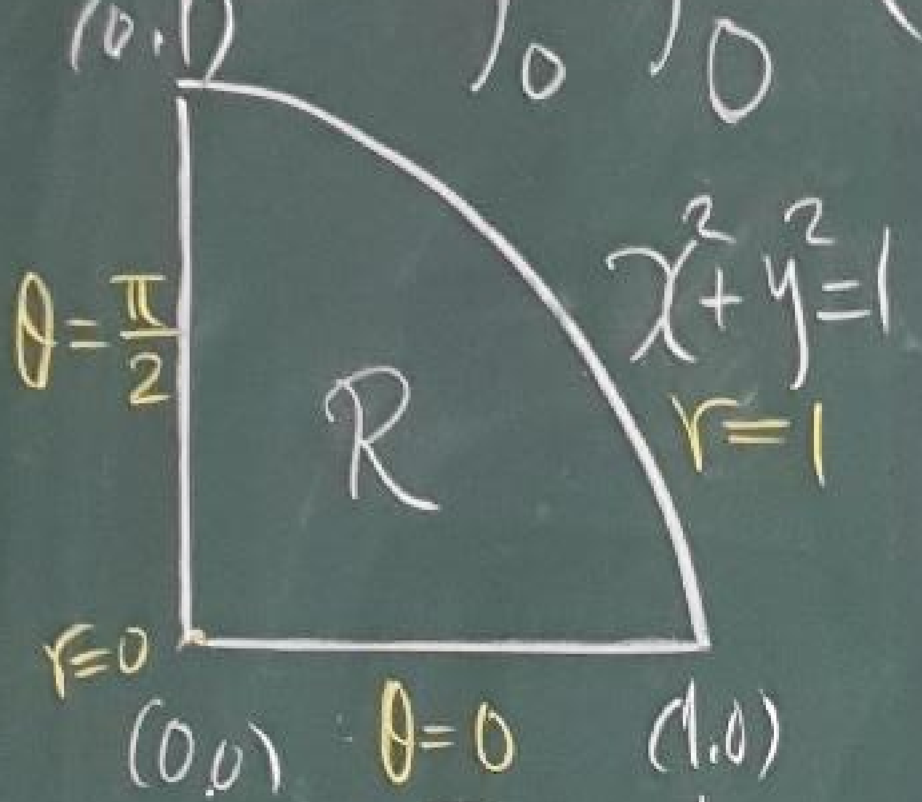
$$I=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\int_{1}^{2} f(r\cos\theta,\ r\sin\theta)\,r\,dr\,d\theta$$



$\Delta A =$  $-$  $= \dfrac{\left(r+\frac{\Delta r}{2}\right)^2 \Delta\theta}{2} - \dfrac{\left(r-\frac{\Delta r}{2}\right)^2 \Delta\theta}{2}$

$$= r\,\Delta r\,\Delta\theta \Longrightarrow dA = r\,dr\,d\theta$$

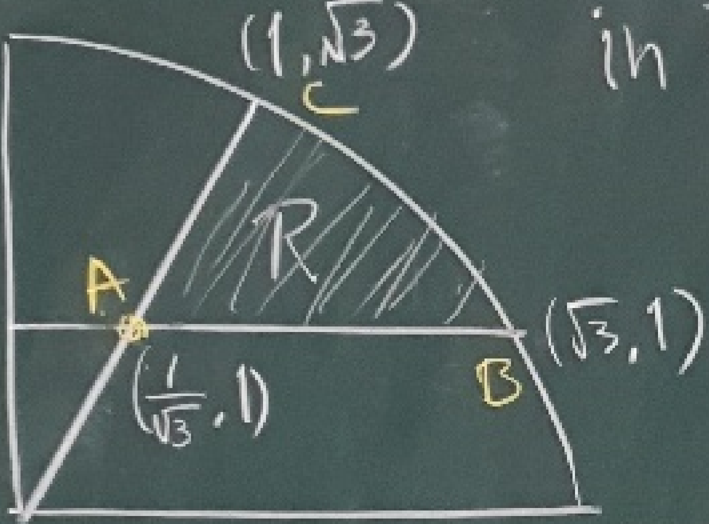
Eg: $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$



$$= \int_0^{\frac{\pi}{2}}\int_0^1 r^2\,\underline{r\,dr\,d\theta}$$

$$= \frac{\pi}{2}\cdot\frac{1}{4} = \frac{\pi}{8}$$

Example: Find area bounded by $y=\sqrt{3}x$, $x^2+y^2=4$, $y=1$ in first quadrant



Method 1:

Area = 
(simple)

Method 2: Area $= \int_{1}^{\sqrt{3}}\int_{\frac{y}{\sqrt{3}}}^{\sqrt{4-y^2}} 1 \, dx \, dy =$

= complicated

From Method 1: A = (pi - sqrt(3))/3

Method 3: (polar coord)

$y=\sqrt{3}x \Leftrightarrow \theta=\frac{\pi}{3}$

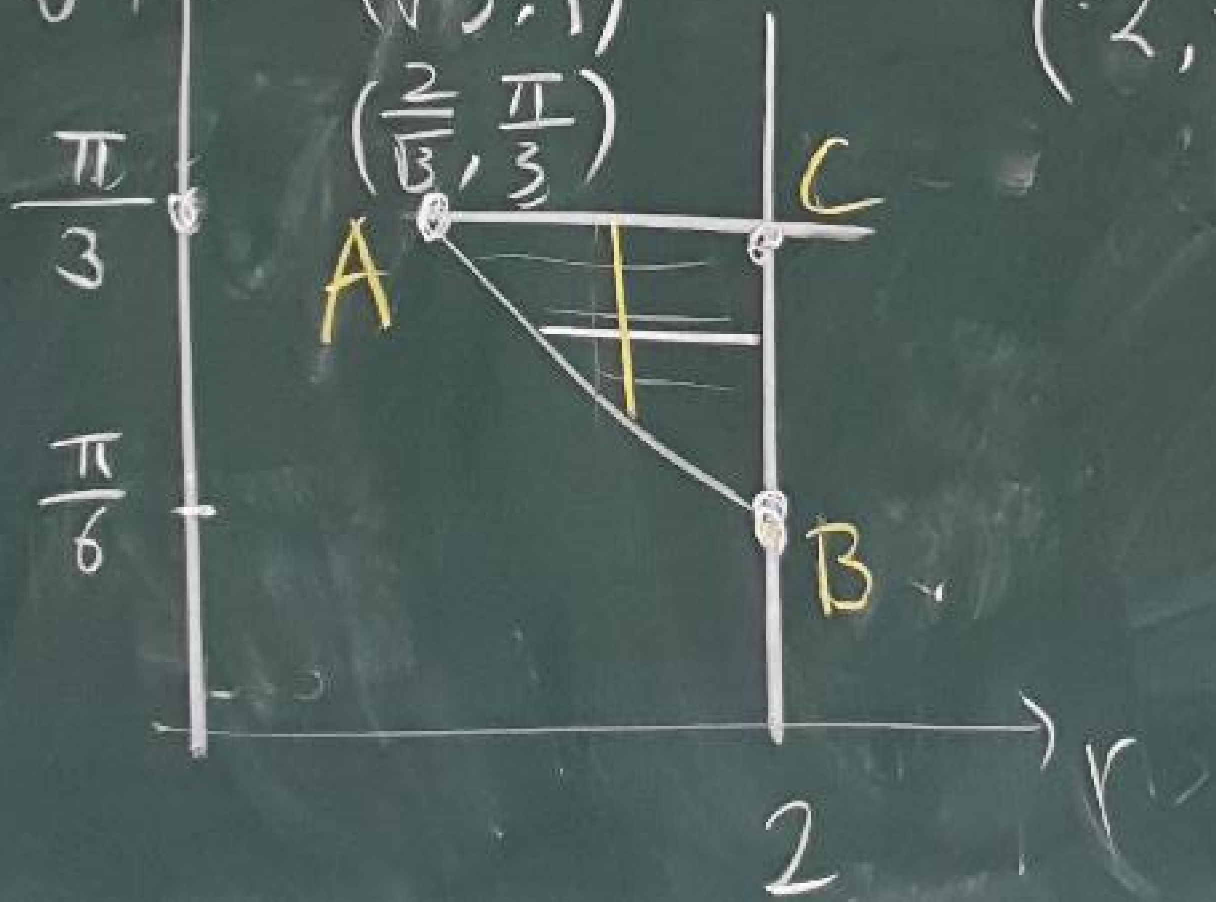
$x^2+y^2=4 \Leftrightarrow r=2$

$r=\csc\theta$

$y=1 \Leftrightarrow \underline{r\sin\theta=1}$ or $\theta=\sin^{-1}\left(\frac{1}{r}\right)$

$(x,y)=\left(\frac{1}{\sqrt{3}},1\right) \Leftrightarrow (r,\theta)=\left(\frac{2}{\sqrt{3}},\frac{\pi}{3}\right)$

$(\sqrt{3},1)$ $\left(2,\frac{\pi}{6}\right)$



$$\text{Area} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_{\csc\theta}^{2} r \, dr \, d\theta \cdots (\mathrm{I})$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{4-\csc^2\theta}{2} \right) d\theta$$

(= ~~more~~ not very complicated) (see textbook for details of integration)

$$\text{or} = \int_{\frac{2}{\sqrt{3}}}^{2} r \int_{\sin^{-1}\left(\frac{1}{r}\right)}^{\frac{\pi}{3}} 1 \, d\theta \, dr \cdots (\mathrm{II})$$

$$= \int_{\frac{2}{\sqrt{3}}}^{2} r \left(\frac{\pi}{3} - \sin^{-1}\left(\frac{1}{r}\right) \right) dr$$

(most complicated)