Partial derivative with Constraint

Example: Find $\frac{\partial w}{\partial \underline{\underline{x}}}$

if $w = x^2 + y^2 + z^2$

and $z = x^2 + y^2$.

$\underline{\text{Rm}}$ 1 eq + 1 constraint = 2 eq

could be

$$\begin{cases} w = w_1(\underline{\underline{x}}, y) \\ z = z(\underline{\underline{x}}, y) \end{cases} \text{ or } \begin{cases} w = w_2(\underline{\underline{x}}, z) \\ y = y(\underline{\underline{x}}, z) \end{cases}$$

$\Rightarrow$ different $\frac{\partial w}{\partial x}$

Need to specify

$$\left(\frac{\partial w}{\partial x}\right)_y \text{ or } \left(\frac{\partial w}{\partial x}\right)_z \quad \text{(different)}$$

$$\begin{array}{lcc} & \downarrow & \downarrow \\ \text{indep. var.}: & (x, y) & (x, z) \end{array}$$

Here

$$\begin{cases} w_1(x, y) = (x^2+y^2) + (x^2+y^2)^2 \\ z(x, y) = x^2+y^2 \end{cases}$$

$$\begin{cases} w_2(x, z) = z + z^2 \\ \\ y(x, z) = \pm\sqrt{z - x^2} \end{cases}$$

Eg. Find $\left(\frac{\partial w}{\partial x}\right)_y$ at $(x, y, z) = (2, -1, 1)$

if $w = x^2 + y^2 + z^2$

and $z^3 - xy + yz + y^3 = 1$

Ans: $w = w(x, y)$, $z = z(x, y)$

$$w_x = 2x + 2z z_x \quad \text{— ①}$$

$$3z^2 z_x - y + y z_x = 0 \quad \text{— ②}$$

$$② \Rightarrow \left(\frac{\partial z}{\partial x}\right)_y (2,-1,1) = \left.\frac{y}{3z^2 + y}\right|_{(2,-1,1)} = \frac{-1}{2}$$

$$\xrightarrow{①} \left(\frac{\partial w}{\partial x}\right)_y (2,-1,1) = 4 + 2 \cdot \frac{-1}{2} = 3$$

Eg. Find $\left(\frac{\partial w}{\partial x}\right)_{y,z}$ if $w = x^2 + y - z + \sin t$

and $x + y = t$

indep. var.: x, y, z

dep. var.: $w(x,y,z)$, $t(x,y,z)$

$$\Rightarrow w(x,y,z) = x^2 + y - z + \sin(x+y)$$

$$\left(\frac{\partial w}{\partial x}\right)_{y,z} = 2x + \cos(x+y).$$

Eg Find $\left(\frac{\partial z}{\partial x}\right)_y$ if

$$f(x, y, z, w) = 0$$

and $g(x, y, z, w) = 0$

Ans: $z = z(x, y)$, $w = w(x, y)$

$$\Rightarrow \left(\frac{\partial}{\partial x}\right)_y \Rightarrow \begin{cases} f_x + f_z z_x + f_w w_x = 0 \\ g_x + g_z z_x + g_w w_x = 0 \end{cases}$$

2 unknowns (z_x, w_x), 2 eqns

$\Rightarrow$ Solve 2×2 linear systems.

Eg: Find $\left(\frac{\partial(PV)}{\partial T}\right)_{n,V}$

if $PV = nRT$, $R = \text{cost}$.

Sol: indep. var. T, n, V

$$P = P(T, n, V)$$

$$\frac{\partial(P(T,n,V)\cdot V)}{\partial T} = P_T V$$

$$P = \frac{nRT}{V}, \quad \left(\frac{\partial P}{\partial T}\right)_{n,V} = \frac{nR}{V}$$

$$\text{Ans} = \left(\frac{\partial P}{\partial T}\right)_{n,V} \cdot V = nR$$

$\underline{\underline{\text{Rm}}}$ $\left(\frac{\partial(PV)}{\partial T}\right)_n$ $PV = nRT$

$\Rightarrow P = f(T, n)$ $V = g(T, n)$

$f(T, n)\, g(T, n) = nRT$

Suprisingly, $\left(\frac{\partial(PV)}{\partial T}\right)_n$

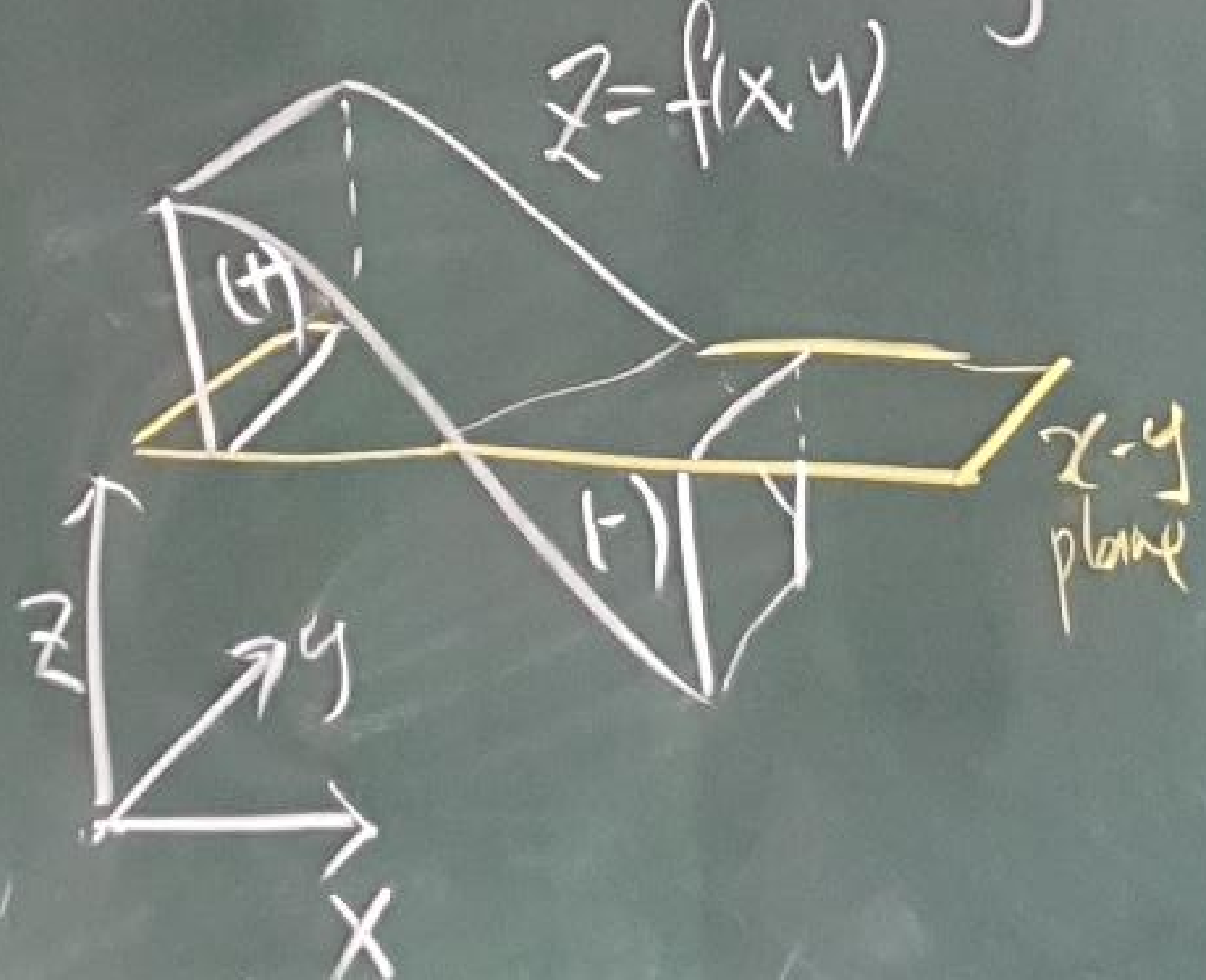
does not depend explicitly

on f or g (HW)

Double integral

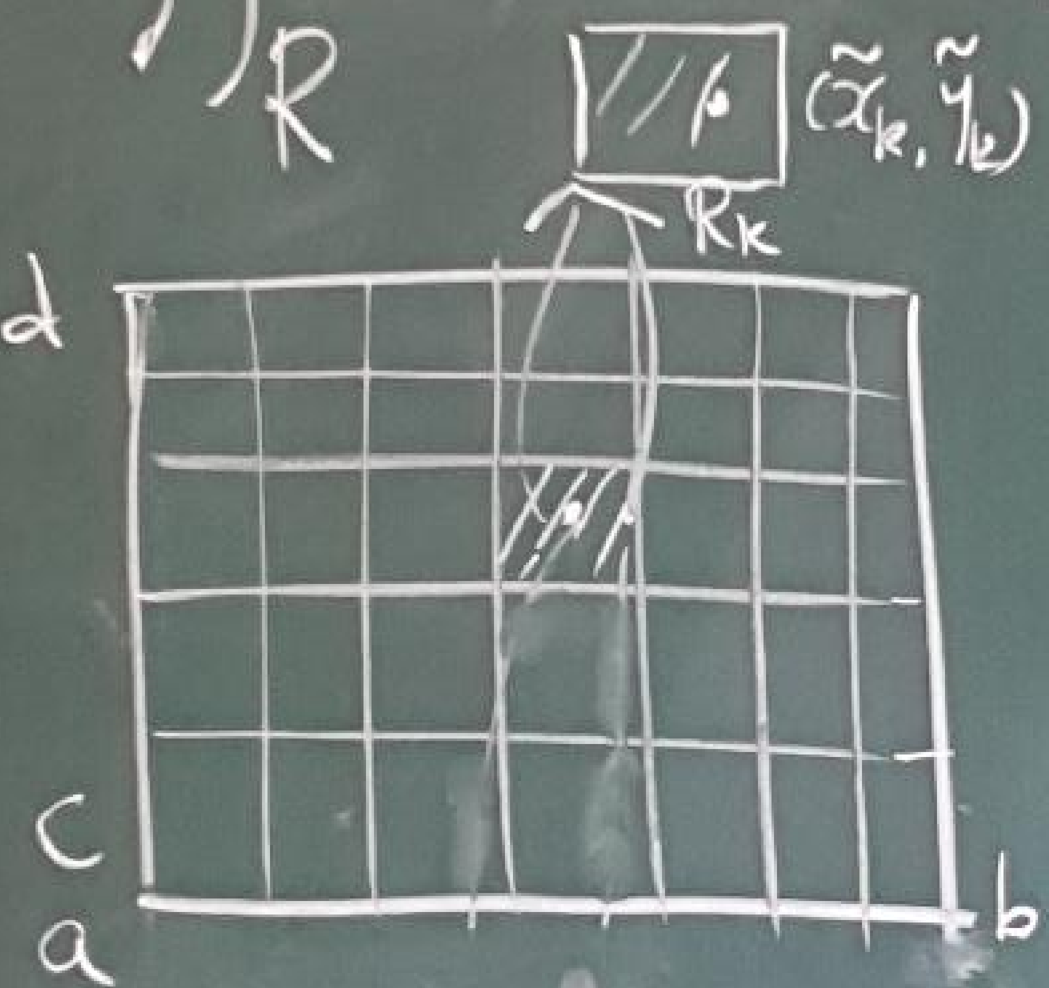
$$\iint_R f(x,y)\,dA \quad R \subseteq \mathbb{R}^2$$

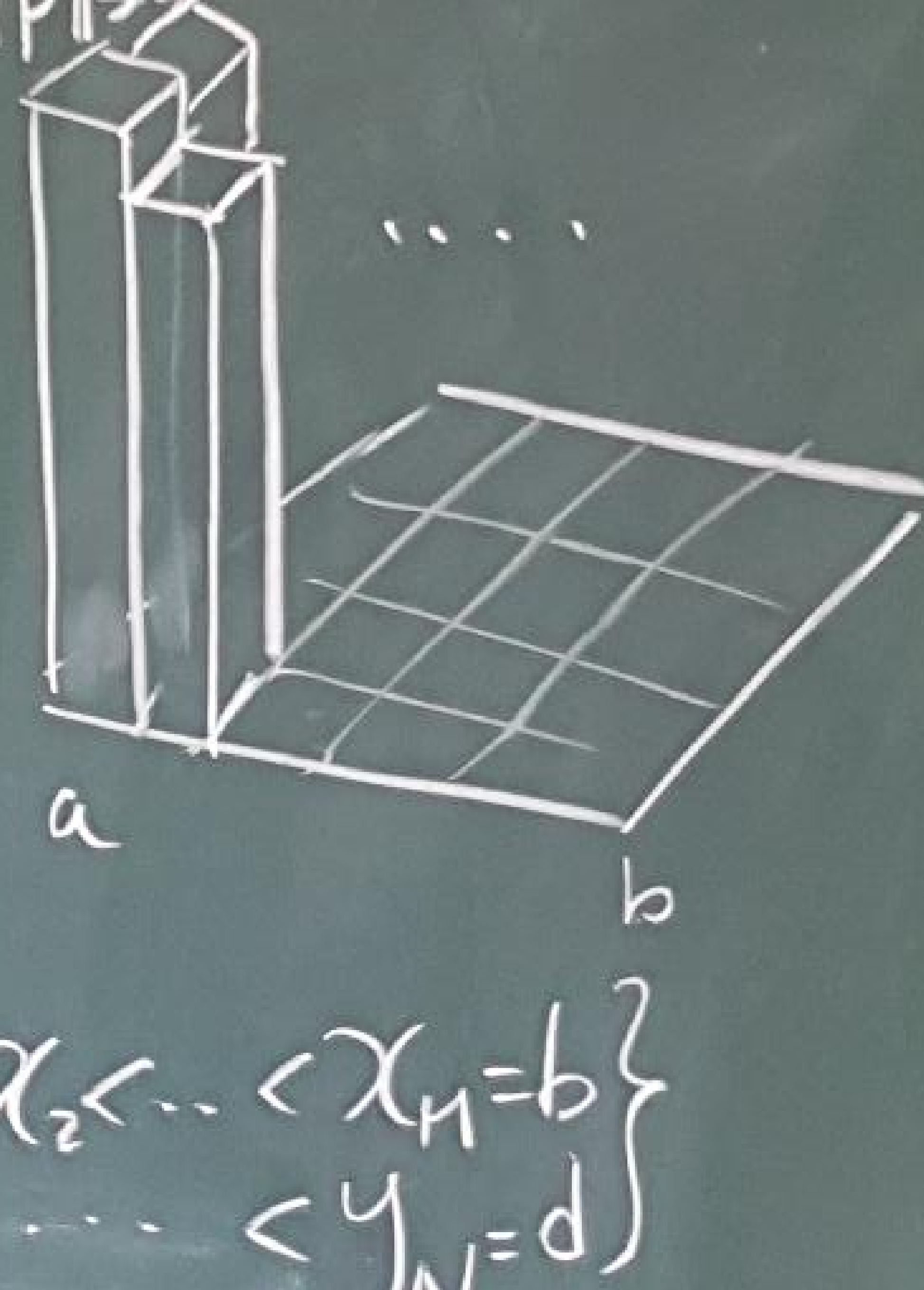
Def Signed volume between $z = f(x,y)$ and x-y plane over the region R



If $R = [a,b] \times [c,d]$

$$\iint_R f(x,y)\,dA = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(\tilde{x}_k, \tilde{y}_k)\,\Delta A_k$$





$$P = \begin{cases} a = x_0 < x_1 < x_2 < \cdots < x_M = b \\ c = y_0 < y_1 < \cdots < y_N = d \end{cases}$$

$$\|P\| = \max_{i,j} \{\Delta x_i, \Delta y_j\}$$

$$\Delta x_i = x_i - x_{i-1},\ \Delta y_j = y_j - y_{j-1},\ \tilde{x}_k, \tilde{y}_k \in R_k$$

Fubini's Thm (1st form)

Let $R = [a,b] \times [c,d]$
and $f(x,y)$ is cont. on R

Then $\iint_R f(x,y)\, dA$

$$= \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$$

Rm We sometimes write

$$\iint_R f(x,y)\, dx\, dy$$

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Rm (i) We sometimes write $\iint_R f(x,y)\,dx\,dy$

(ii) If $f(x,y)$ is cont. R then $\iint_R f(x,y)\,dA$ exists

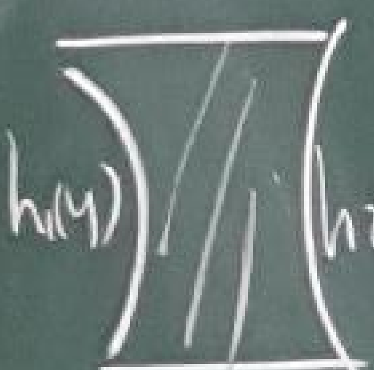
We say f is integrable over R

Fubini's Thm: If f is cont. on R

(a) If $R = \{a \le x \le b \ \ g_1(x) \le y \le g_2(x)\}$

then $\displaystyle\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$

(b) If $R = \{c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$



then $\displaystyle\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$