Eg. find the nearest point to origin on $E: \left(x-\frac{1}{2}\right)^2+\frac{y^2}{4}=1$

Sol: $$\underset{x,y\in E}{\text{minimize}} \sqrt{x^2+y^2}$$

$$\Longleftrightarrow \underset{x,y\in E}{\text{minimize}}\ x^2+y^2$$

$$\Longleftrightarrow \text{minimize } f(x,y)=x^2+y^2$$

subjet to $g(x,y)=\left(x-\frac{1}{2}\right)^2+\frac{y^2}{4}-1=0$

$$1: \left(x-\frac{1}{2}\right)^2+\frac{y^2}{4}=1 \quad (g=0)$$

$$2: 2x=\lambda 2\left(x-\frac{1}{2}\right) \quad (f_x=\lambda g_x)$$

$$3: 2y=\lambda\frac{y}{2} \quad (f_y=\lambda g_y)$$

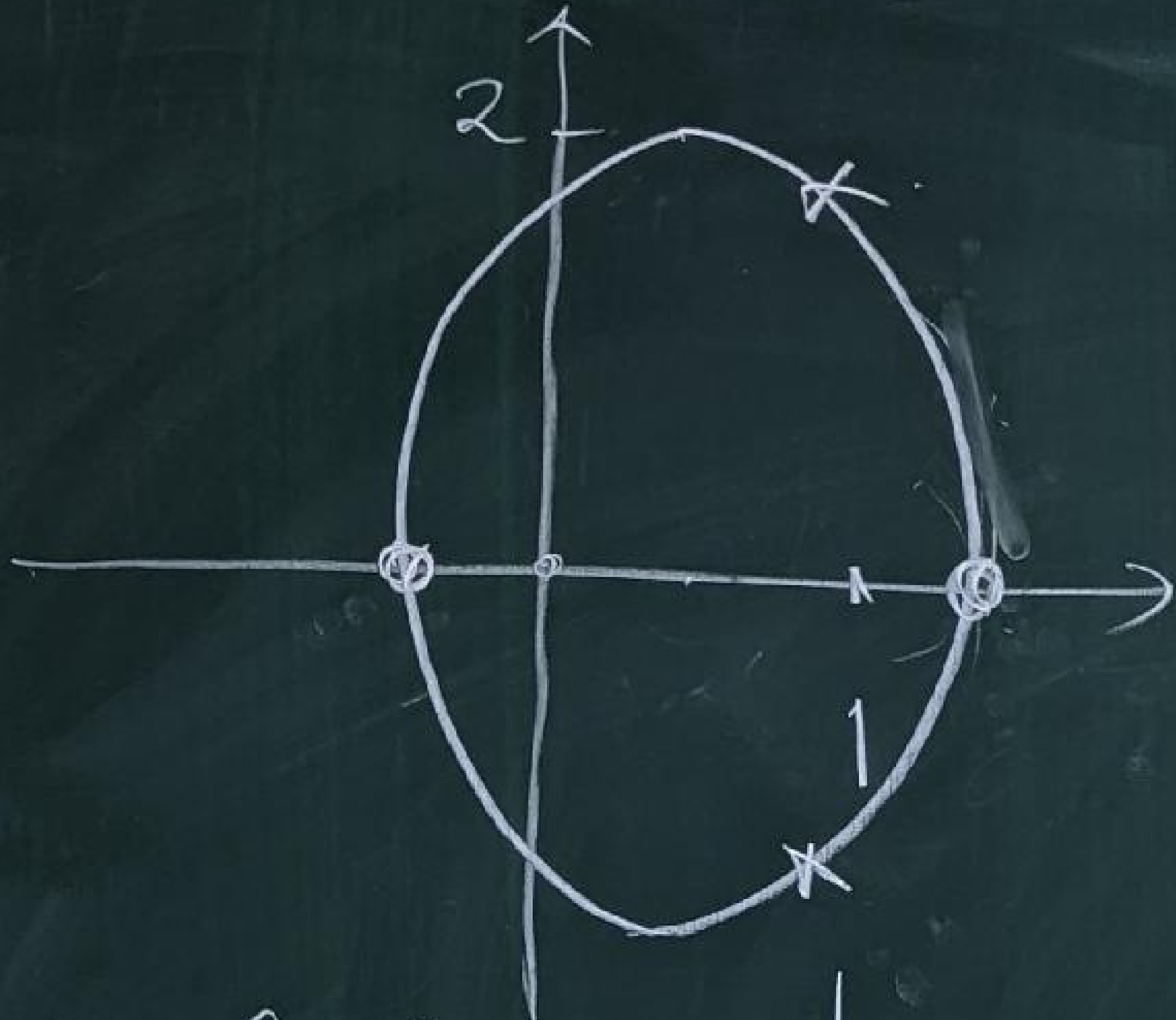
$$3 \Rightarrow \underset{a}{\lambda=4} \text{ or } \underset{b}{y=0}$$

$$\text{Case } a \xRightarrow{2} 2x=8x-4 \Rightarrow x=\frac{2}{3}$$

$$\xRightarrow{1} y=\pm\frac{\sqrt{35}}{3} \Rightarrow f(x,y)=\frac{39}{9}$$

$$\text{Case } b \xRightarrow{1} x=\frac{3}{2} \xrightarrow{2} \lambda=\frac{3}{2} \Rightarrow f(x,y)=\frac{9}{4}$$

$$\text{or } x=-\frac{1}{2} \xrightarrow{2} \lambda=\frac{1}{2} \Rightarrow f(x,y)=\frac{1}{4}$$



Ans: $f\left(\frac{-1}{2}, 0\right) = \frac{1}{4} = \text{min}$

$f\left(\frac{3}{2}, 0\right) = \frac{9}{4} = \text{local min}$

$f\left(\frac{2}{3}, \frac{\pm\sqrt{35}}{3}\right) = \frac{39}{9} = \text{max}$

Similar formulation

for "optimize $f(x,y,z)$

subject to $g(x,y,z)=0$"

4 unknowns: x, y, z, λ

4 eqs: $g=0$, $\nabla f=\lambda \nabla g$

or "optimize $f(x,y,z,w)$

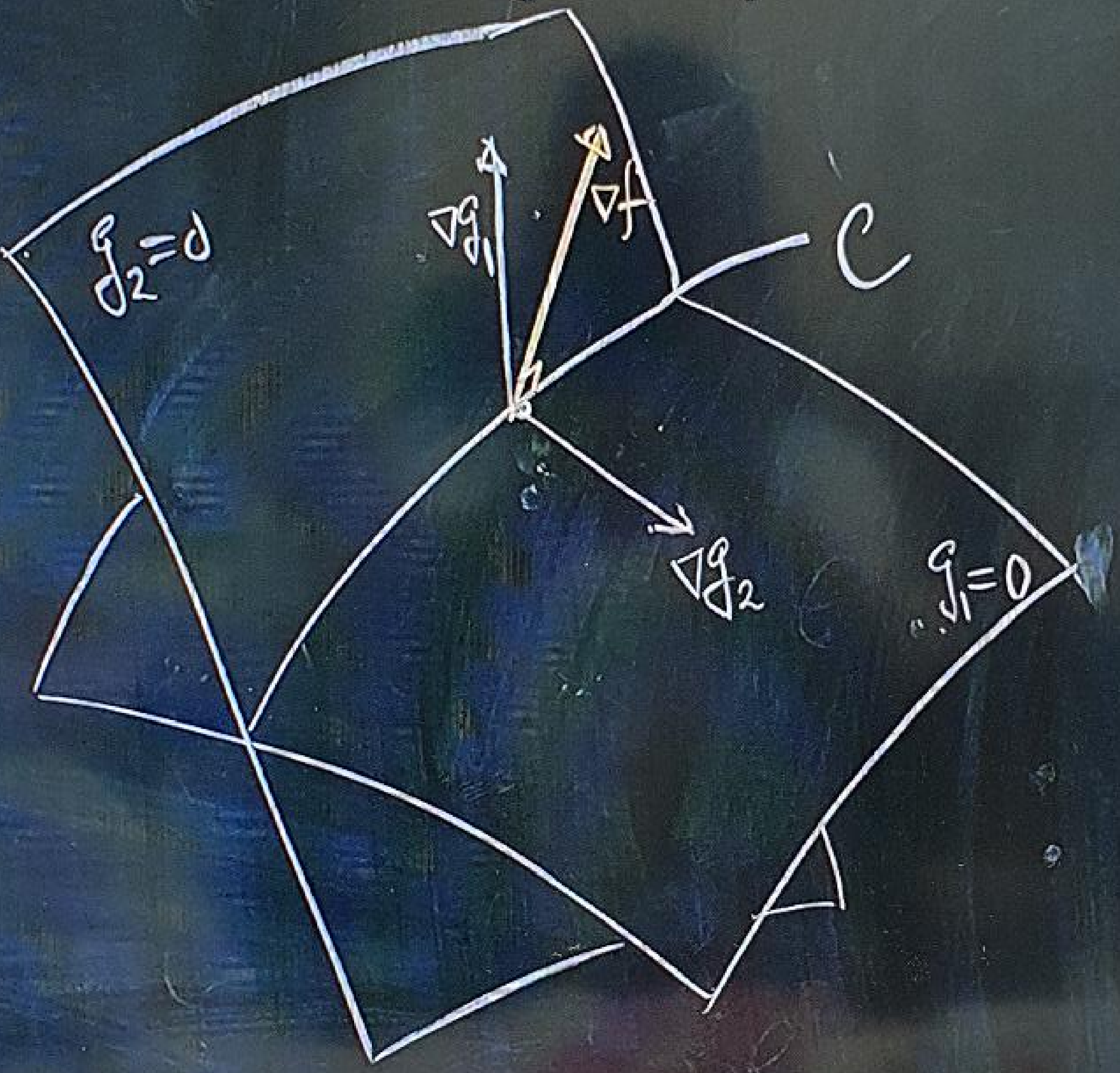
subject to $g(x,y,z,w)=0$"

5 unknowns

5 equations

§ Optimize $f(x, y, z)$

Subject to $\begin{cases} g_1(x, y, z) = 0 \\ g_2(x, y, z) = 0 \end{cases} \cdots (C)$



At a critical point (x_0, y_0, z_0)

C is tangent to level surface of f : $\{(x,y,z) \mid f(x,y,z) = f(x_0,y_0,z_0)\}$

$\Longrightarrow \nabla f(x_0,y_0,z_0) \perp C$

Moreover: $\nabla g_1(x_0,y_0,z_0) \perp C$

$\nabla g_2(x_0,y_0,z_0) \perp C$

$\Longrightarrow \nabla f, \nabla g_1, \nabla g_2$ are coplane at (x_0,y_0,z_0)

$$\begin{cases} \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 (x_0,y_0,z_0) \\ g_1(x_0,y_0,z_0) = 0 \\ g_2(x_0,y_0,z_0) = 0 \end{cases}$$

5 unknowns, 5 eqns

Eg: Find nearest point

on $\begin{cases} g_1(x,y,z) = x+y+z = 1 \\ g_2(x,y,z) = x^2+y^2 = 1 \end{cases}$ to $(0,0,0)$

<u>Sol</u> $f(x,y,z) = x^2+y^2+z^2$

$$\begin{cases} x+y+z = 1 & \text{—— ①} \\ x^2+y^2 = 1 & \text{—— ②} \\ 2x = \lambda_1 1 + \lambda_2 2x & \text{— ③} \\ 2y = \lambda_1 1 + \lambda_2 2y & \text{— ④} \\ 2z = \lambda_1 1 + \lambda_2 \cdot 0 & \text{— ⑤} \end{cases}$$

$$③④ \xrightarrow{⑤} \begin{aligned} x(1-\lambda_2) &= z \\ y(1-\lambda_2) &= z \end{aligned}$$

$\Rightarrow$ ⓐ : $\lambda_2 = 1,\ z = 0$

or ⓑ : $x = y = \dfrac{z}{1-\lambda_2}$

Case ⓐ:

$③, ④ \xrightarrow{ⓐ} \lambda_1 = 0$

$① ② \xrightarrow{ⓐ} (x, y) = (1, 0), (0, 1)$

$(x, y, z) = (1, 0, 0)$ or $(0, 1, 0)$

Case ⓑ: $② \xrightarrow{ⓑ} (x, y) = \pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\xrightarrow{①} z = 1 \mp \sqrt{2}$

(λ_1, λ_2 are not important)

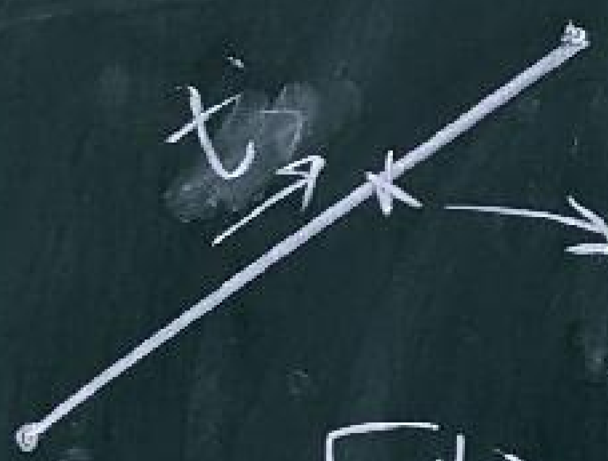
Crit. pts $(1,0,0)$, $(0,1,0)$, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1-\sqrt{2}\right)$, $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 1+\sqrt{2}\right)$

$f(x,y,z)$ $\quad 1, \quad 1, \quad 1+(1-\sqrt{2})^2, \quad 1+(1+\sqrt{2})^2$

min $\quad$ min $\quad$ (local max ?) $\quad$ max

Taylor formula for $f(x,y)$:

Center: (a,b), $x = a+h$, $y = b+k$



$(x,y) = (a+h, b+k)$

$(a+th, b+tk)$, $t \in [0,1]$

(a,b) $\quad F(t) = f(a+th, b+tk)$ $\quad t \in [0,1]$

Assume all partial derivatives of f of order up to $(n+1)$ are continuous

(*) $\Rightarrow F(t)$ is $(n+1)$ times cont.-diff. in t
(see below)

$\Rightarrow$ Taylor formula for $F(t)$ holds

$$F(1) = F(0) + 1 \cdot F_t(0) + \frac{1}{2} F_{tt}(0) + \cdots + \frac{F^{(n)}(0)}{n!} 1^n + \frac{F^{(n+1)}(c)}{(n+1)!} 1^{n+1}$$

$$F'(t) = (h\partial_x + k\partial_y) f(a+th, b+tk)$$

$$F''(t) = (h\partial_x + k\partial_y) \frac{d}{dt} f(a+th, b+tk) = (h\partial_x + k\partial_y)^2 f(a+th, b+tk)$$

cont.-diff. = 'continuously differentiable', meaning the derivative is continous.

(*) means: all partial derivatves of order up to (n+1) are continous.

where $(h\partial_x + k\partial_y)^2 = h^2\partial_x^2 + 2hk\,\partial_x\partial_y + k^2\partial_y^2$

Similarly

$$\frac{d^\ell}{dt^\ell} f(a+th,\, b+tk) = (h\partial_x + k\partial_y)^\ell f$$

$$f(x,y) = F(1) = F(0) + F'(0) + \cdots + \frac{F^{(n+1)}(c)}{(n+1)!}$$

$$= f(a,b) + (h\partial_x + k\partial_y) f(a,b) + \frac{(h\partial_x + k\partial_y)^2}{2} f(a,b)$$

$$+ \cdots + \frac{(h\partial_x + k\partial_y)^n}{n!} f(a,b) + \frac{(h\partial_x + k\partial_y)^{n+1}}{(n+1)!} f(a+ch,\, b+ck)$$

Similarly $f(x,y,z) = \displaystyle\sum_{k=0}^{n} \frac{\left((x-x_0)\partial_x + (y-y_0)\partial_y + (z-z_0)\partial_z\right)^k}{k!} f(x,y,z)$

$$+ \frac{(\cdots\cdots)^{n+1}}{(n+1)!} f(x_0 + c(x-x_0),\, y_0 + c(y-y_0) + z_0 + c(z-z_0))$$