Remark: $\Delta f = \Delta_1 + \Delta_2 + \Delta_3$

$\Delta_1 = \nabla f(x_0, y_0) \cdot (\Delta x, \Delta y)$

$\Delta_2 = \frac{1}{2}\left(A(\Delta x)^2 + 2B\Delta x\Delta y + C(\Delta y)^2\right)$

$A = f_{xx}(x_0, y_0),\ B = f_{xy},\ C = f_{yy}$

$\Delta_3 = O\left(\Delta x^3, \Delta x^2\Delta y, \Delta x\Delta y^2, \Delta y^3\right)$

At a critical point, $\nabla f = \vec{0}$

(if f is diff.) $\Rightarrow \Delta_1 = 0$

Δ_2:

(i) $A>0,\ B^2-AC<0 \Rightarrow \Delta_2>0$

(ii) $A<0,\ B^2-AC<0 \Rightarrow \Delta_2<0$

(iii) If $B^2-AC>0$

Then $\Delta_2=0 \Leftrightarrow A\left(\frac{\Delta x}{\Delta y}\right)^2+2B\frac{\Delta x}{\Delta y}+C=0$

$$\Leftrightarrow \frac{\Delta x}{\Delta y} = \frac{-B\pm\sqrt{B^2-AC}}{2A}$$

$(-B-\sqrt{\ })/2A$ $\qquad$ $(-B+\sqrt{\ })/2A$ $\qquad \longrightarrow \frac{\Delta x}{\Delta y}$

$A>0$:	$\Delta_2>0$	$\Delta_2<0$	$\Delta_2>0$
$A<0$:	$\Delta_2<0$	$\Delta_2>0$	$\Delta_2<0$

How to find local extrema of $f(x,y)$ (diff.) on $\mathbb{R}^2$?

<u>Step 1</u>: Find all critical points (a,b) with $\nabla f(a,b)=\vec{0}$

<u>Step 2</u>: $D=\left(f_{xy}^2-f_{xx}f_{yy}\right)(a,b)$

$(f_{xx}, D) \Rightarrow \begin{cases} \text{local min} \\ \text{local max} \\ \text{saddle point} \end{cases}$

Eg: find local extrema of $f(x,y) = x^2 + y^2 - 4y + 9$

Sol: $f_x = 2x$, $f_y = 2y - 4$

$\nabla f = \vec{0} \Longleftrightarrow (x_0, y_0) = (0,2)$

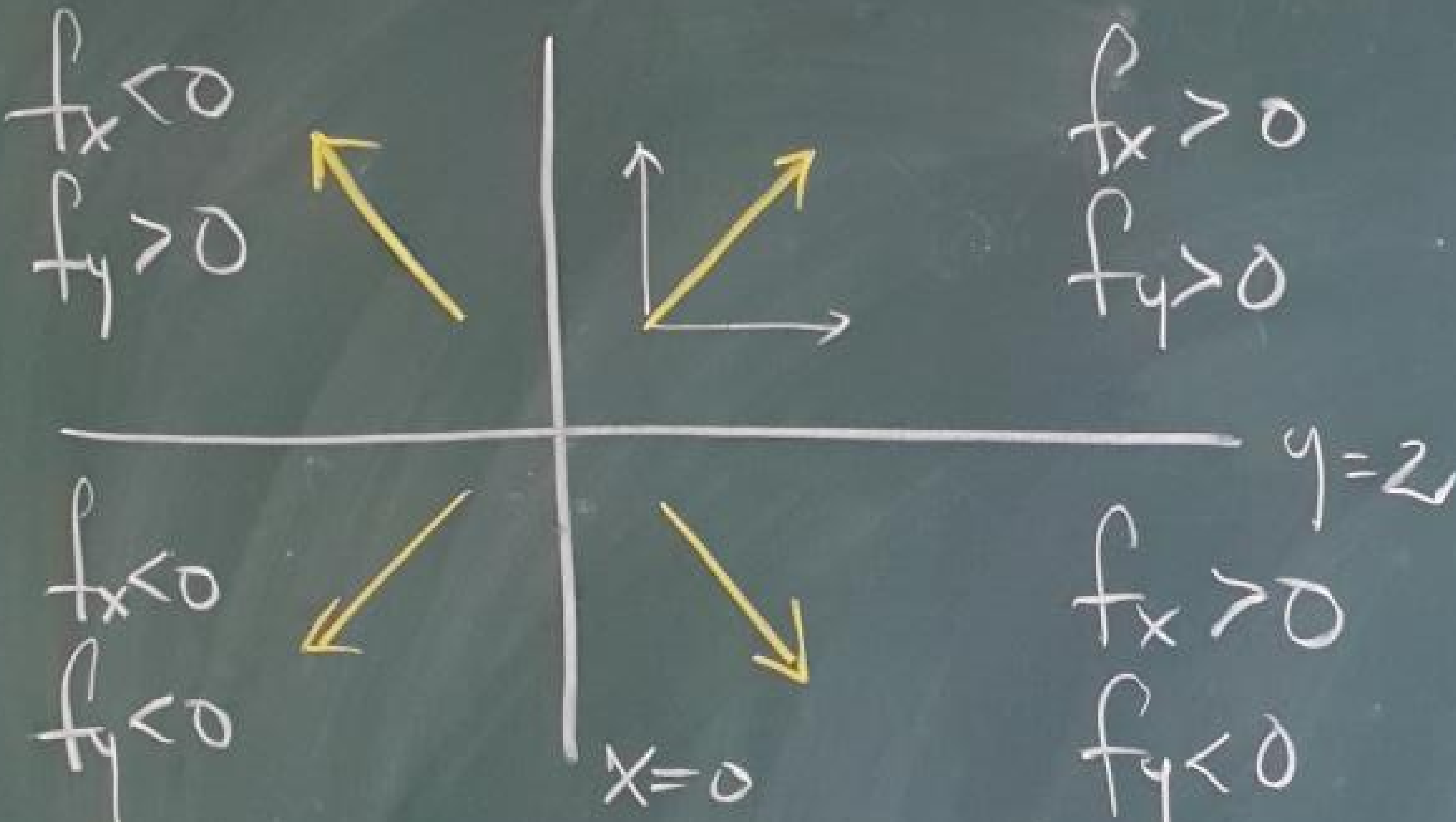
$f_{xx}(0,2) = 2$, $f_{xy}(0,2) = 0$, $f_{yy} = 2$

$f_{xx} > 0$, $f_{xy}^2 - f_{xx} f_{yy} < 0$

Only local extreme point $= (0,2)$

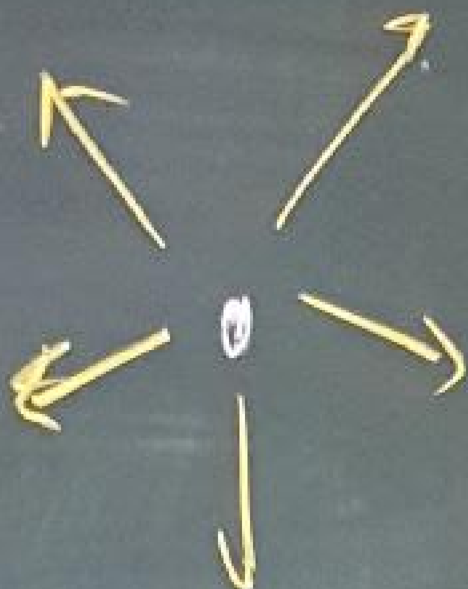
$=$ local min.

method 2: (gradient analysis)

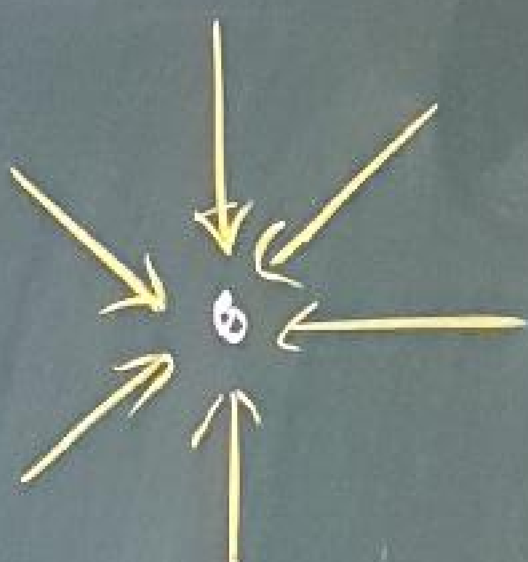


<u>Rm</u> Gradient analysis is applicable even if $A=0$ or $D=0$ (analogue of $f(x)=x^3$ in 1D)

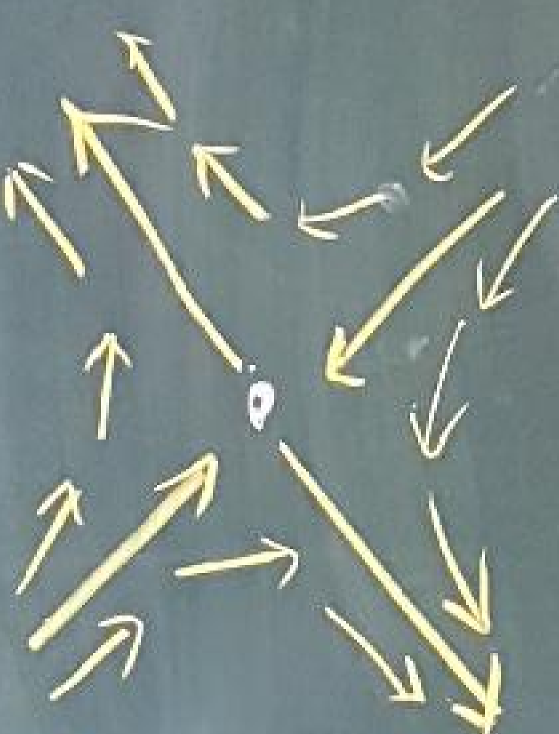
$\underline{\mathbb{R}^m}$



local min



local max



Saddle point

Eg Find abs. min/max

of $f(x,y) = 2 + 2x + 2y - x^2 - y^2$

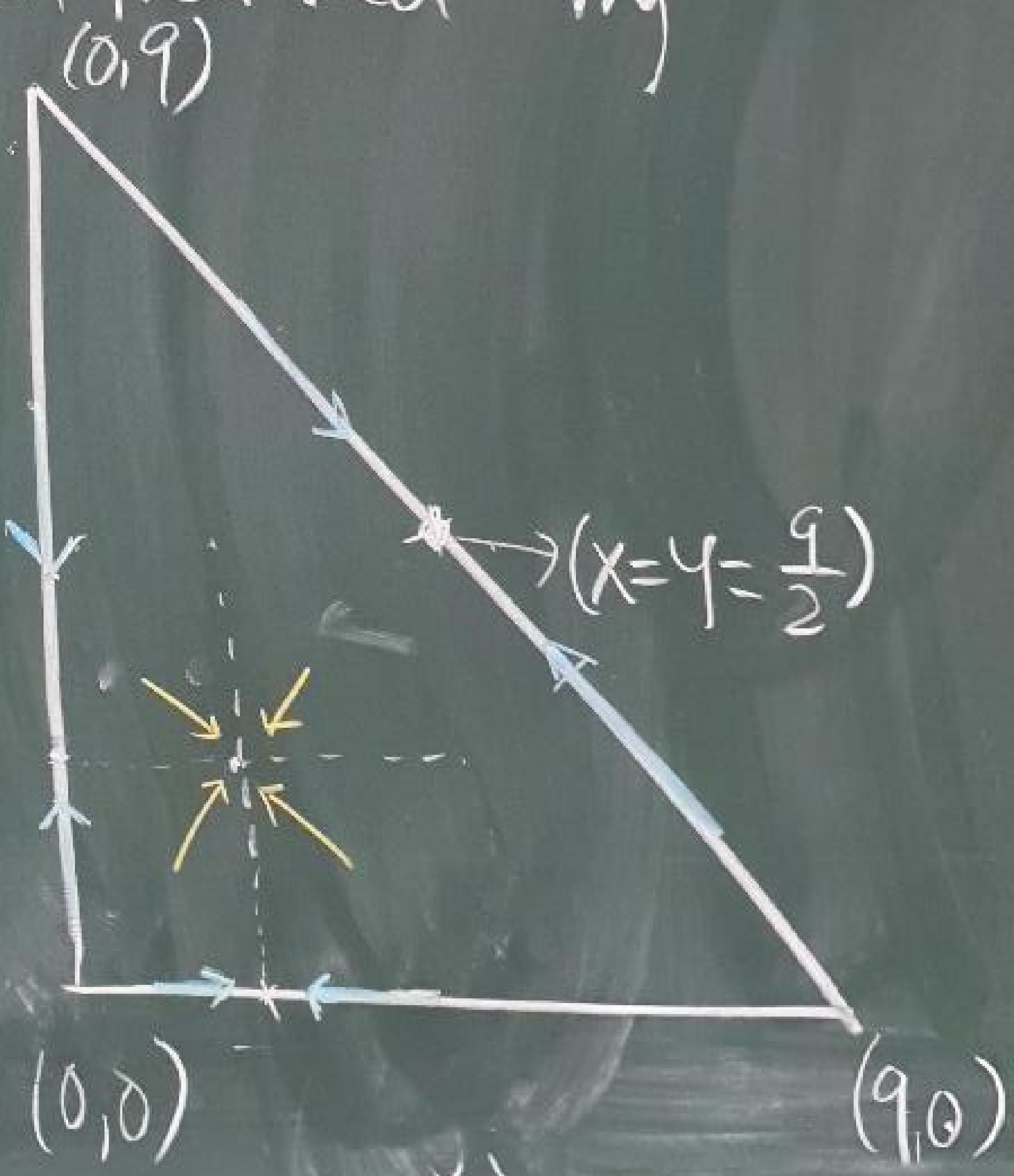
on the region bounded by

$$\begin{cases} x = 0 \\ y = 0 \\ y = 9 - x \end{cases}$$

$f_x = -2(x-1)$

$f_y = -2(y-1)$

$\nabla f = \vec{0} \iff (1,1)$



Step 1: Plot ∇f at interior

Step 2: Plot tangential component of ∇f at boundary

Here on $x=0$ and $y=0$: obvious

on $x+y=9$:

plot $\nabla f \cdot \vec{t}$, $\vec{t} = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$= -2(x-1, y-1)\cdot(-1,1)/\sqrt{2}$$

$$= \sqrt{2}(x-y) \begin{cases} >0 & x>y \\ <0 & x<y \end{cases}$$

)

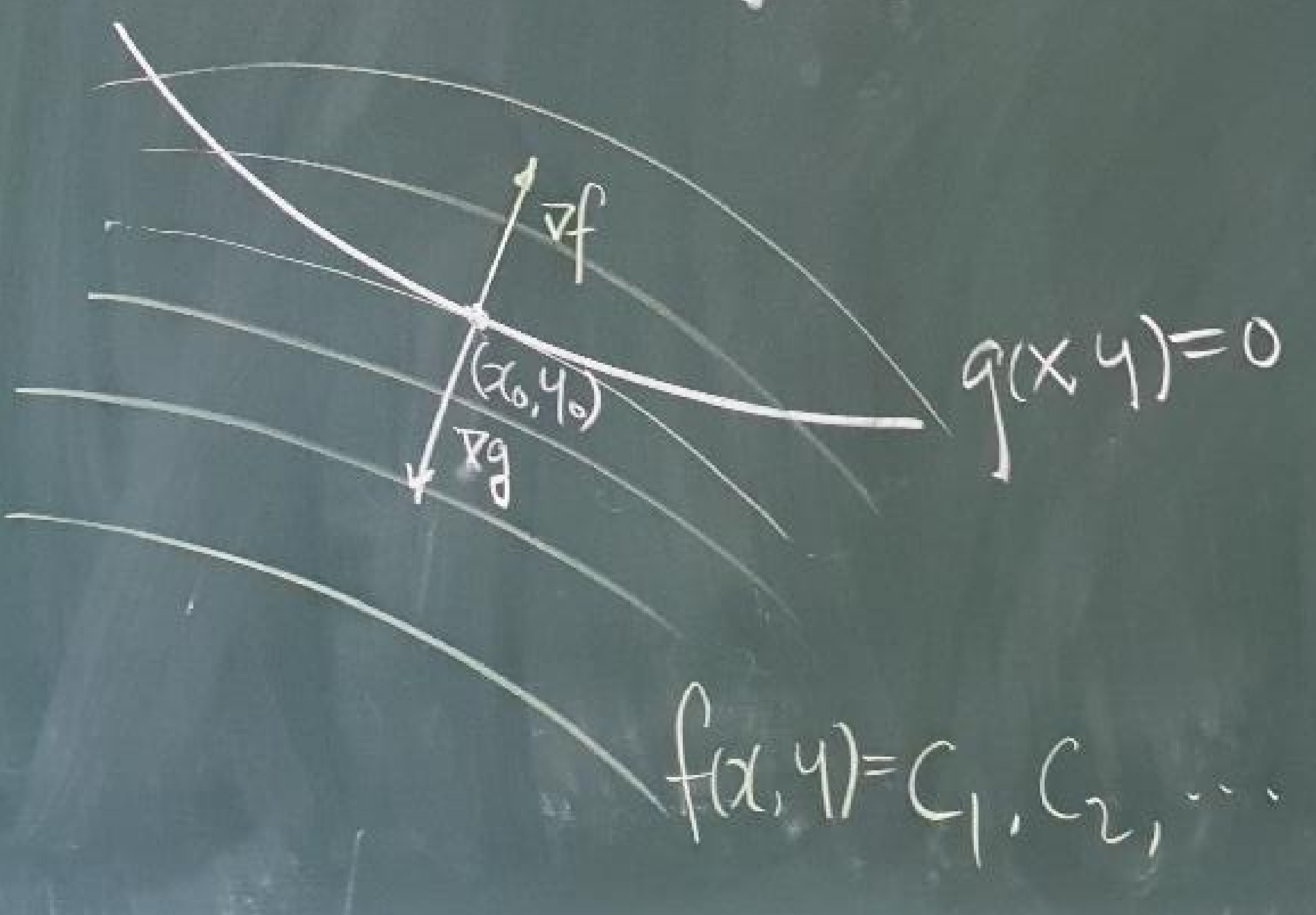
Gradient analysis

$$\Rightarrow \begin{cases} f(1,1) = \text{local max} = \text{abs max} \\ f(0,0), f(9,0), f(0,9) = \text{local min} \end{cases}$$

$$\underset{2}{\overset{\shortparallel}{f(0,0)}} \qquad \underbrace{\underset{-61}{\overset{\shortparallel}{f(9,0)}} \quad \underset{-61}{\overset{\shortparallel}{f(0,9)}}}_{\text{abs min}}$$

Constrained optimization

"Find local extreme points of $f(x,y)$ subject to the constraint $g(x,y)=0$"



It suffices to find all (x_0, y_0) where

$\nabla f(x_0, y_0) \parallel \nabla g(x_0, y_0)$

$$\begin{cases} g(x_0, y_0) = 0 \\ \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \end{cases}$$

3 unknowns: x_0, y_0, λ

3 equations: $\underset{g=0}{1} + \underset{\nabla f = \lambda \nabla g}{2}$