If f_x and f_y are cont. in R

$(x_0, y_0) \in R$ (R is open)

$\Rightarrow f$ is diff. at (x_0, y_0)

$\Rightarrow f =$ "linearization" $+$ error

(error is smaller than r.)

Question: What is the usual form of error?

Thm If, $f, f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ are all cont. in an open region R, $(x_0, y_0) \in R$

and $|f_{xx}|, |f_{xy}|, |f_{yy}| \leq M$ on R, then

$$|f(x,y) - L(x,y)| \leq \frac{M}{2}(|x-x_0| + |y-y_0|)^2$$

Rm: In fact

$$f(x,y) - L(x,y) = \frac{1}{2}\left(\begin{array}{l} f_{xx}(c,d)(x-x_0)^2 + f_{yy}(c,d)(y-y_0)^2 \\ + 2f_{xy}(c,d)(x-x_0)(y-y_0) \end{array}\right),$$

where (c,d) lies on the segment "$\overline{(x_0,y_0)\ (x,y)}$"

Extreme Values and Saddle points

How to find local min/max of $f(x,y)$?

(1): If $\nabla f(x_0,y_0) \neq (0,0)$

$$\Rightarrow \Delta f = \underline{\nabla f(x_0,y_0)\cdot(\Delta x,\Delta y)} + o\left(\sqrt{\Delta x^2+\Delta y^2}\right)$$

$(\Delta x,\Delta y) \parallel \nabla f$

$(\Delta x,\Delta y) \parallel -\nabla f$

$$= \begin{cases} |\nabla f|\cdot|(\Delta x,\Delta y)| + o(\,\cdots) \\ -|\nabla f|\cdot|(\Delta x,\Delta y)| + o(\,\cdots) \end{cases}$$

$|(\Delta x,\Delta y)| = \sqrt{\Delta x^2+\Delta y^2}$

$$\| \qquad \underline{(\text{linear part})} + \left(\begin{matrix}\text{error}\\ (\text{neglectible})\end{matrix}\right)$$

$\Rightarrow$ Not local min/max

Since $\Delta f>0$ in one direction and <0 in another direction.

Here $\Delta f = f(x,y) - f(x_0,y_0)$

$\therefore$ f has local min/max at (x_0, y_0)

$$\begin{aligned} &\Longrightarrow \\ &\not\Longleftarrow \end{aligned} \quad \nabla f(x_0, y_0) = (0,0)$$

If $\nabla f(x_0, y_0) = (0,0)$

$$\Rightarrow \Delta f = \frac{1}{2}\begin{pmatrix} f_{xx}(c,d)(\Delta x)^2 + f_{yy}(c,d)(\Delta y)^2 \\ + 2 f_{xy}(c,d)\, \Delta x\, \Delta y \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} f_{xx}(x_0,y_0)(\Delta x)^2 + f_{yy}(x_0,y_0)\, \Delta y^2 \\ + 2 f_{xy}(x_0,y_0)\, \Delta x\, \Delta y \end{pmatrix} + \Delta_3$$

where $\Delta_3 = O(\Delta x)^3 + O(\Delta x^2 \Delta y) + O(\Delta x \Delta y^2) + O(\Delta y^3)$

$\therefore$ at a critical point $\left(\nabla f(x_0,y_0)=(0,0)\right)$

$$\Delta f = \frac{1}{2}\Big(\underbrace{A\Delta x^2 + B\Delta x\Delta y + C\Delta y^2}_{\Delta_2}\Big)$$

$+$ Smaller term

where $A=f_{xx}(x_0,y_0)$, $B=f_{xy}(x_0,y_0)$, $C=f_{yy}(x_0,y_0)$

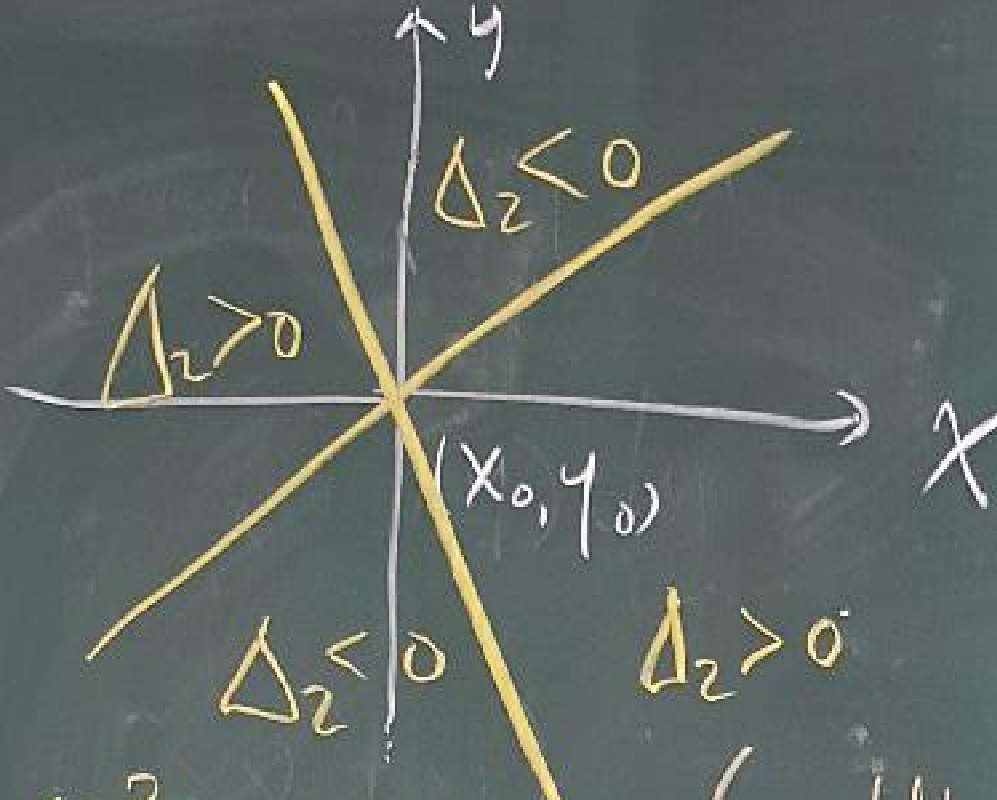
$$\Delta_2 = \frac{1}{2}\left(A\Delta x^2 + B\Delta x\Delta y + C\Delta y^2\right)$$

$(\Delta x,\Delta y)\neq(0,0)$

$$\Delta_2>0 \iff \begin{matrix} A>0 \\ B^2-4AC<0 \end{matrix}$$ local min

$$\Delta_2<0 \iff A<0,\ B^2-4AC<0$$ local max

$$\Delta_2 \begin{matrix} >0 \\ <0 \end{matrix} \iff B^2-4AC>0$$ saddle point



$$B^2 - 4AC > 0$$ (saddle point)