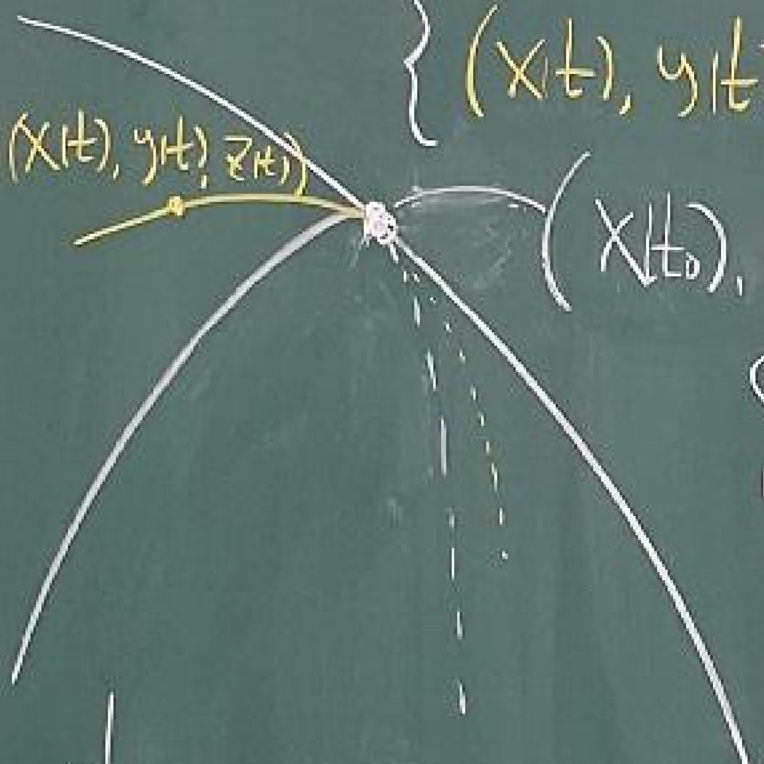
Tangent planes and normal lines

Let $S=\{(x,y,z)\mid F(x,y,z)=C\}$

the level surface of $F:\mathbb{R}^3\longrightarrow\mathbb{R}$ (diff.)

$\{(x(t),y(t),z(t))\;|t-t_0|<\delta\}\subseteq S$

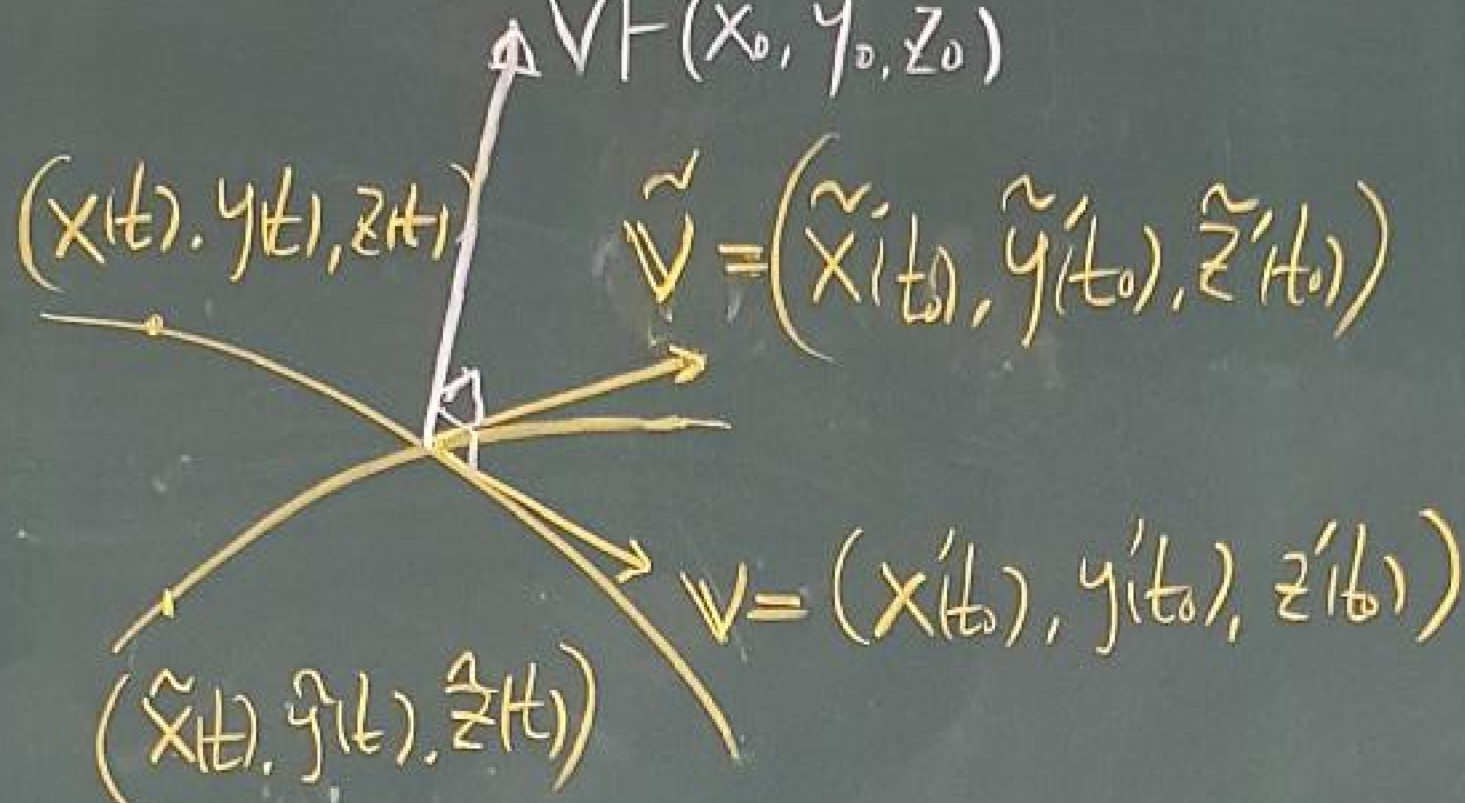


$(x(t_0),y(t_0),z(t_0))=(x_0,y_0,z_0)\in S$

S = level surface:

$F(x(t),y(t),z(t))\equiv C$

$\left.\dfrac{d}{dt}\right|_{t=t_0}\Rightarrow\nabla F(x(t_0),y(t_0),z(t_0))\cdot(x'(t_0),y'(t_0),z'(t_0))=0$



Since $(x'(t_0), y'(t_0), z'(t_0))$ = tangent vector

$\Rightarrow \nabla F(x_0, y_0, z_0) \perp$ (any tangent vector at (x_0, y_0, z_0))

$\Rightarrow \nabla F(x_0, y_0, z_0) \perp$ (tangent plane at (x_0, y_0, z_0))

$\Rightarrow$ Normal vector $/\!/ \; \nabla F(x_0, y_0, z_0)$

Normal line: $\dfrac{x - x_0}{F_x(x_0, y_0, z_0)} = \dfrac{y - y_0}{F_y} = \dfrac{z - z_0}{F_z}$

Also: tangent plane

$$\nabla F(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

A special case:

If $F(x, y, z) - C = f(x, y) - z$

Then $\nabla F(x, y, z) = (f_x(x, y), f_y(x, y), -1)$

tangent plane in this case reduces to

$$\underset{\substack{\| \\ f(x_0, y_0, z_0)}}{z - z_0} = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Rm: parametrization of normal line:

$x = x_0 + t F_x(x_0, y_0, z_0)$, $y = y_0 + t F_y$, $z = z_0 + t F_z$

Eg: Find the tangent plane and normal line of

$$x^2+\frac{y^2}{4}+\frac{z^2}{9}=3 \text{ at } (1,2,3)$$

<u>Sol</u>: $F(x,y,z)$ $\left(= x^2+\frac{y^2}{4}+\frac{z^2}{9}\right)$

$$\nabla F=\left(2x,\frac{y}{2},\frac{2z}{9}\right)$$

$$\therefore \nabla F(1,2,3)=\left(2,1,\frac{2}{3}\right)$$

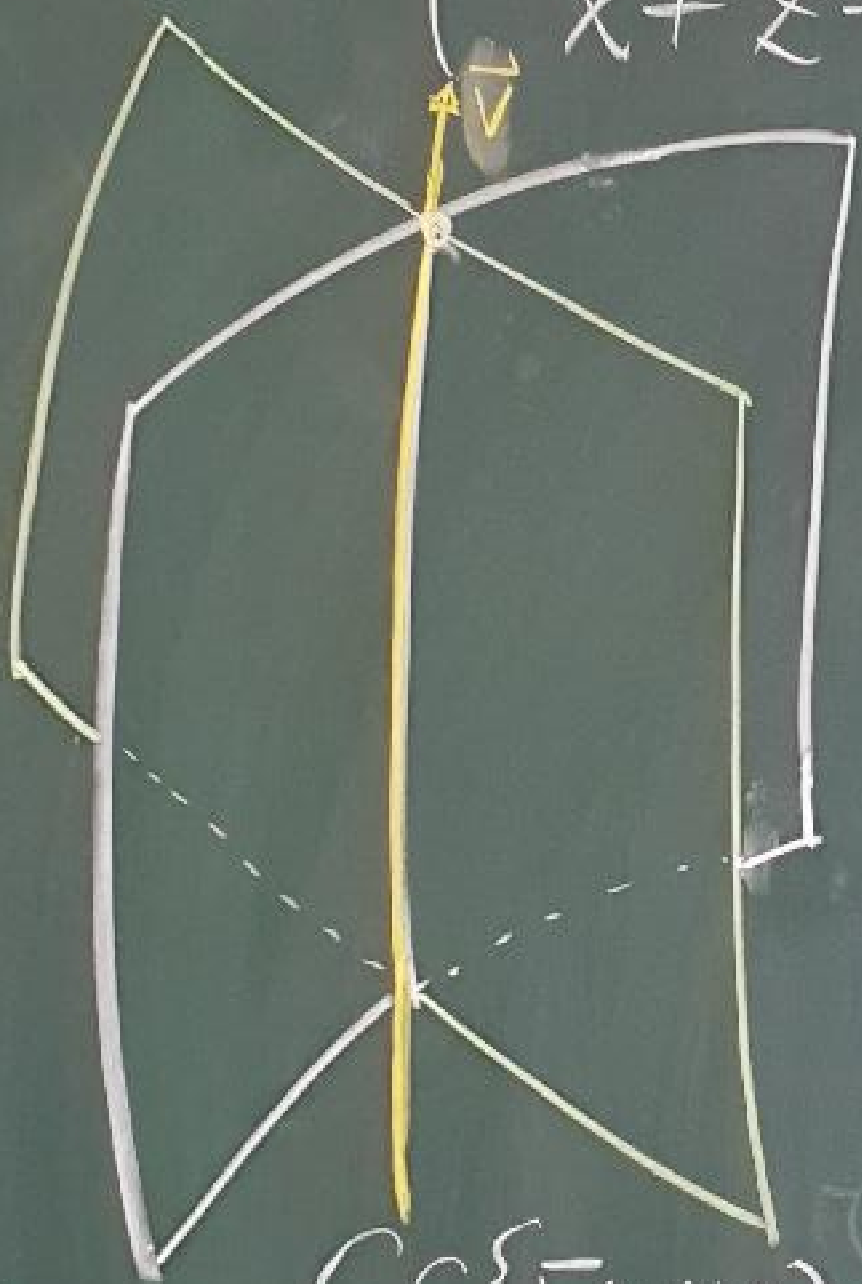
tangent plane:

$$\left(2,1,\frac{2}{3}\right)\cdot(x-1,y-2,z-3)=0$$

Normal line: $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2/3}$

or: $x=1+2t,\ y=2+t,\ z=3+\frac{2}{3}t$

Eg: Find tangent line of the curve $\begin{cases} x^2+y^2-2=0 \\ x+z-4=0 \end{cases}$ at $(1,1,3)$



$C: \begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$

$(x_0,y_0,z_0)\in C$

$\vec{v}$ = tangent vector of C at (x_0,y_0,z_0)

$C \subseteq \{F(x,y,z)=0\} \Rightarrow \vec{v} \perp \nabla F(x_0,y_0,z_0)$

$C \subseteq \{G(x,y,z)=0\} \Rightarrow \vec{v} \perp \nabla G(x_0,y_0,z_0)$

$\Rightarrow \vec{t} \parallel \nabla F(x_0,y_0,z_0) \times \nabla G(x_0,y_0,z_0)$

Here $F = x^2+y^2-2$, $G = x+z-4$

$\nabla F(1,1,3) = (2x, 2y, 0)_{(1,1,3)} = (2,2,0)$

$\nabla G(1,1,3) = (1,0,1)$

tangent line

$(x-1, y-1, z-3) \parallel \nabla F \times \nabla G$

$= \begin{vmatrix} i & j & k \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = (2,-2,-2)$

i.e. $x = 1+2t$, $y = 1-2t$, $z = 3-2t$

) Normal plane:

$(x-1, y-1, z-3)\cdot(2,-2,-2) = 0$

Linearization and differential

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linear approximation

= approximating $z = f(x, y)$

by a linear function $z = L(x, y)$

(near $(x_0, y_0, f(x_0, y_0))$)

linear function:

$$L(x, y) = ax + by + c$$

$$= a(x - x_0) + b(y - y_0) + f(x_0, y_0)$$

linear approximation near $(x_0, y_0, f(x_0, y_0))$ means:

$$f(x, y) - L(x, y) = o\left(\sqrt{(x - x_0)^2 + (y - y_0)^2}\right)$$

Note that

$$o\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right)=o(x-x_0)+o(y-y_0)$$

$$=\varepsilon_1(x-x_0)+\varepsilon_2(y-y_0),\ \lim_{(x,y)\to(x_0,y_0)}(\varepsilon_1,\varepsilon_2)=(0,0)$$

Conclusion: $z=L(x,y)$ is the linearization of $z=f(x,y)$ at (x_0,y_0)

$$\Leftrightarrow f(x,y)-\underset{\substack{\| \\ a(x-x_0)+b(y-y_0)+f(x_0,y_0)}}{L(x,y)}=\varepsilon_1(x-x_0)+\varepsilon_2(y-y_0)$$

Note: Let $y=y_0$, $x\to x_0$

$$f(x,y_0)-\big(f(x_0,y_0)+a(x-x_0)\big)=\varepsilon_1(x-x_0)$$

$$\lim_{x\to x_0}\frac{1}{x-x_0}(\cdots)\Rightarrow a=f_x(x_0,y_0)\ \left(\text{Similarly, } b=f_y(x_0,y_0)\right)$$

Conclusion:

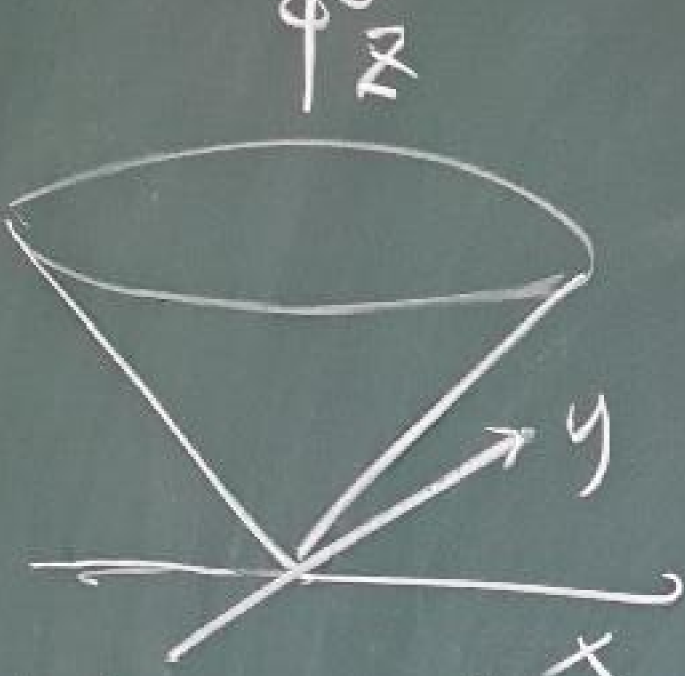
$z = f(x, y)$ has a linearization near (x_0, y_0)

$\Leftrightarrow$ $f(x, y)$ is diff. at (x_0, y_0)

Moreover, the linearization is uniquely given by

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \qquad (*)$$

Eg: Does $z = f(x,y) = \sqrt{x^2+y^2}$ have a tangent plane at (x_0, y_0)?



Method 1: take any $L(x,y) = ax + by$ and show that

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2+y^2}} \neq 0$$

In fact, two path Thm $\Rightarrow$ limit does not exist

Method 2

Note: $f_x(0,0)$ does not exist

$$f_x(0^+,0) = 1 \neq -1 = f_x(0^-,0)$$

$\Rightarrow$ The only candidate of linearization (*) does not exist!

$\therefore$ tangent plane does not exist

($f(x,y)$ is NOT diff. at $(0,0)$)