Implicit Diff. Revisited.

Eg Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(0,0,0)$

if $z(x,y)$ is implicitly defined

by $\underbrace{x^3 + z^2 + y e^{xz} + z\cos y}_{F(x,y,z)} = 0$

$F(x,y,z)=0$

Sol $F(x,y,z(x,y)) = 0$

$\Rightarrow \frac{\partial}{\partial x} F(x,y,z(x,y)) = 0$

$$0 = \partial_x F(x, y, z(x,y))$$

$$= \underbrace{F_x(x,y,z)}_{\text{fixed } y,z} \frac{\partial x}{\partial x} + \underbrace{F_z(x,y,z)}_{\text{fixed } x,y} \frac{\partial z}{\partial x}$$

$$= F_x(x,y,z)\Big|_{z=z(x,y)} + F_z(x,y,z)\Big|_{z=z(x,y)} \frac{\partial z}{\partial x}(x,y)$$

$$= F_x + F_z \frac{\partial z}{\partial x}$$

$$\therefore \frac{\partial z}{\partial x}\Big|_{(x_0,y_0,z_0)} = -\frac{F_x(x_0,y_0,z_0)}{F_z(x_0,y_0,z_0)}$$

Similarly $\displaystyle \frac{\partial z}{\partial y}\Big|_{(x_0,y_0,z_0)} = -\frac{F_y(x_0,y_0,z_0)}{F_z(x_0,y_0,z_0)}$

$$F_x(0,0,0) = 3x^2 + yze^{zx} + 9\Big|_{(0,0,0)} = 0$$

$$F_y(0,0,0) = e^{xz} - z\sin y\Big|_{(0,0,0)} = 1$$

$$F_z(0,0,0) = 2z + xye^{xz} + \cos y\Big|_{(0,0,0)} = 1$$

$$\frac{\partial z}{\partial x}\Big|_{(0,0,0)} = -\frac{F_x(0,0,0)}{F_z(0,0,0)} = 0$$

$$\frac{\partial z}{\partial y}\Big|_{(0,0,0)} = -\frac{F_y(0,0,0)}{F_z(0,0,0)} = -1$$

Directional derivative.

Def: "Derivative of $f(x,y)$ at $P_0 = (x_0, y_0)$ in the direction of unit vector $\vec{u} = (u_1, u_2)$" (i.e. $u_1^2 + u_2^2 = 1$)

$$\left(\frac{df}{ds}\right)_{\vec{u}, P_0} = D_{\vec{u}} f(x_0, y_0)$$

$$= \lim_{\Delta s \to 0} \frac{f(x_0 + \Delta s\, u_1, y_0 + \Delta s\, u_2) - f(x_0, y_0)}{\Delta s}$$

<u>Thm</u> If $f(x,y)$ is diff. at (x_0,y_0), then

$$D_{\vec{u}} f(x_0,y_0) = \nabla f(x_0,y_0)\cdot\vec{u}$$

where $\nabla f = (f_x, f_y)$.

<u>pf</u>: $$\Delta f = f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \quad (\lim \varepsilon_1, \varepsilon_2 = 0)$$

$$\Delta x = \Delta s\, u_1, \quad \Delta y = \Delta s\, u_2$$

$$\lim_{\Delta s \to 0} \frac{\Delta f}{\Delta s} = f_x(x_0,y_0)\,u_1 + f_y(x_0,y_0)\,u_2 = \nabla f(x_0,y_0)\cdot\vec{u}$$

Eg. $f(x,y) = x^2 + xy$

$\vec{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. $\left(\begin{array}{l}\text{Note:} \\ f \text{ is diff.}\end{array}\right)$

$D_{\vec{u}} f(1,2) = ?$

__Sol__ $\displaystyle \lim_{h\to 0} \frac{f\left(1+\frac{h}{\sqrt{2}}, 2+\frac{h}{\sqrt{2}}\right) - f(1,2)}{h}$

$= \frac{d}{dh} f\left(1+\frac{h}{\sqrt{2}}, 2+\frac{h}{\sqrt{2}}\right)\Big|_{h=0}$

$= f_x \frac{1}{\sqrt{2}} + f_y \cdot \frac{1}{\sqrt{2}}$

$= (2x+y, x)\big|_{(1,2)} \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{5}{\sqrt{2}}$

$\underline{\text{Eg}}$. $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$D_{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)} f(0,0) = ?$

$f_x(0,0) = \lim_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h} = 0$

$f_y(0,0) = \lim_{h \to 0} \dfrac{f(0,h) - f(0,0)}{h} = 0$

$D_{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)} f(0,0) = \lim_{h \to 0} \dfrac{f\left(\frac{h}{\sqrt{2}}, \frac{h}{\sqrt{2}}\right) - f(0,0)}{h}$

$= \lim_{h \to 0} \dfrac{\left(\frac{h}{\sqrt{2}}\right)^3 / h^2}{h} = \dfrac{1}{2\sqrt{2}} \neq \nabla f(0,0) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

<u>Remark</u>: In polar coordinate

$$f(r,\theta) = r\cos^2\theta \sin\theta$$

In contrast: a plane passing through $(0,0)$

$$L(x,y) = a(x-0) + b(y-0)$$
$$= r(a\cos\theta + b\sin\theta)$$
$$= L_x(x-0) + L_y(y-0)$$

Properties of ∇f:

If f is diff.

$\Rightarrow D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$

$= |\nabla f(x_0, y_0)| \cdot |\vec{u}| \cdot \cos\theta$

$\Rightarrow$ (1) f increases (decreases) most rapidly in the direction $\vec{u}$ if $\cos\theta = 1$ ($\cos\theta = -1$)

i.e. the direction of $\nabla f(x_0, y_0)$ ($-\nabla f(x_0, y_0)$)

(2): $D_{\vec{u}} f(x_0, y_0) = 0$ if $\nabla f(x_0, y_0) \perp \vec{u}$

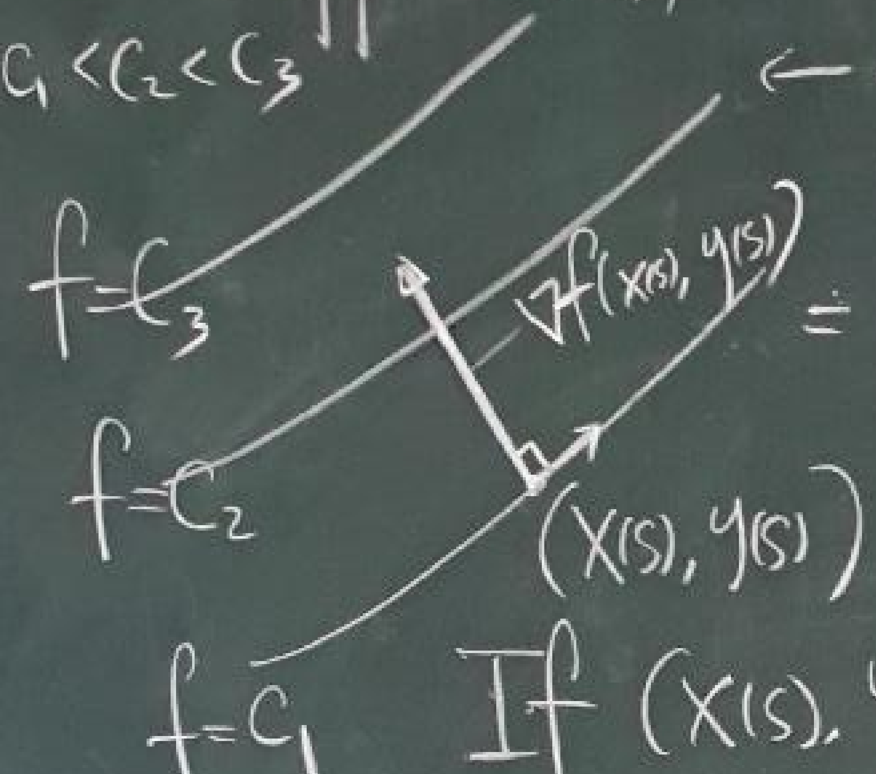
$a < c$

$f = c$

$f =$

Application

$c_1 < c_2 < c_3$



$\leftarrow f(x,y)=\text{const}$

$\doteq \left\{ (x,y) \,\middle|\, f(x,y)=c \right\}$

If $(x(s),y(s))$ is a parametrization of a level curve, then

$$f(x(s),y(s)) = \text{constant}$$

If $f(x,y)$ is diff.

$$\Rightarrow 0 = \frac{d}{ds} f(x(s),y(s)) = \nabla f(x(s),y(s)) \cdot \overbrace{(x'(s),y'(s))}^{\left((x'(s),y'(s)) \text{ is a tangent vector of the level curve}\right)}$$

$$\Rightarrow \nabla f \perp (\text{tangent line of level curve})$$

Eg: Find the tangent line and normal line of

$$\frac{x^2}{4}+y^2=2 \text{ at } (-2,1)$$

Sol: Let $F(x,y)=\frac{x^2}{4}+y^2$

$$\nabla F(-2,1)=\left(\frac{x}{2},2y\right)\Big|_{(-2,1)}=(-1,2)$$

$\therefore$ a normal vector $=(-1,2)$

tangent line: $(x-x_0,y-y_0)\cdot$ normal vector $=0$

i.e. $(x+2,y-1)\cdot(-1,2)=0$; Normal line: $\frac{y-1}{x+2}=\frac{2}{-1}$