Recall. $f'(x_0)$ exists

$$\Leftrightarrow \lim_{x \to x_0} \frac{f(x) - \left(f(x_0) + f'(x_0)(x - x_0)\right)}{x - x_0} = 0$$

Let $(*) = f(x) - \left(f(x_0) + f'(x_0)(x - x_0)\right)$

$$(*) = \begin{cases} \varepsilon\,(x - x_0). \quad \lim_{x \to x_0} \varepsilon = 0 \\ o(1)\,(x - x_0) \\ o(x - x_0) \end{cases}$$

$$\Leftrightarrow f(x) = \left(f(x_0) + f'(x_0)(x - x_0)\right) + (*)$$

Def $f(x,y)$ is diff. at (x_0, y_0)

if $\exists\ a, b, c \in \mathbb{R}$

such that

$$f(x,y) - \left(a(x-x_0) + b(y-y_0) + c\right) = (**)$$

$$(**) = \begin{cases} \text{"} \lim\limits_{(x,y)\to(x_0,y_0)} \dfrac{(**)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0 \text{"} \\ \varepsilon \sqrt{(x-x_0)^2 + (y-y_0)^2}, \ \lim\limits_{(x,y)\to(x_0,y_0)} \varepsilon = 0 \\ o(1) \sqrt{(x-x_0)^2 + (y-y_0)^2} \\ o\left(\sqrt{(x-x_0)^2 + (y-y_0)^2}\right) \end{cases}$$

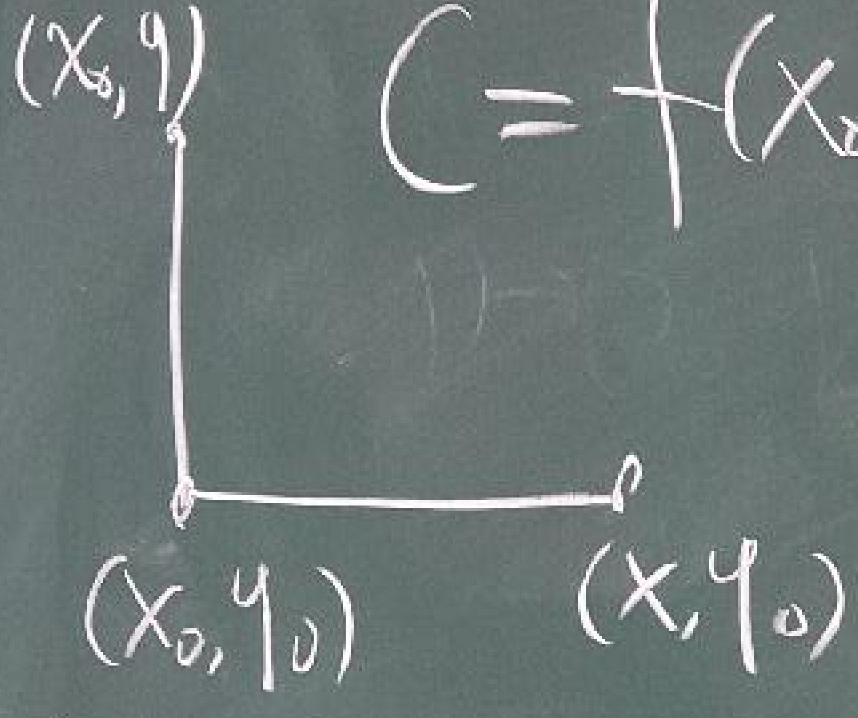
Remark:

(1). If $a, b, c \in \mathbb{R}$ exist

then $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ exist.

and $a = f_x(x_0, y_0)$, $b = f_y(x_0, y_0)$

$c = f(x_0, y_0)$



For example: $f_x(x_0, y_0)$

$$\lim_{x \to x_0} f(x, y_0) - (a(x - x_0) + c) = o(x - x_0)$$

$$\Rightarrow c = f(x_0, y_0),\ a = \lim \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$$

(2)

$$(**) = \varepsilon_1(x-x_0)+\varepsilon_2(y-y_0)$$

$$\lim_{(x,y)\to(x_0,y_0)} \varepsilon_1, \varepsilon_2 = 0.$$

i.e. $(**) = o(x-x_0)+o(y-y_0)$

<u>Def</u> $f(x,y)$ is diff at (x_0,y_0)

if $f_x(x_0,y_0)$, $f_y(x_0,y_0)$ exist

and $f(x,y) - \left(f(x_0,y_0) \begin{array}{l} + f_x(x_0,y_0)(x-x_0) \\ + f_y(x_0,y_0)(y-y_0) \end{array}\right)$

$= \varepsilon_1(x-x_0)+\varepsilon_2(y-y_0)$, $\lim \varepsilon_1, \varepsilon_2 = 0$

$f'(x_0)$ exists

$\Longleftrightarrow$ f is diff. at (x_0)

$\Longleftrightarrow$ $f(x)$ can be appx.

by $y = \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{L(x)}$

near $(x_0, f(x_0))$

$\Longleftrightarrow$ $f(x) - L(x) = o(x - x_0)$

$\left(\text{or} = \varepsilon (x - x_0),\ \lim\limits_{x \to x_0} \varepsilon = 0\right)$

$\Longleftrightarrow$ Existence of tangent line

$f_x(x_0, y_0)$ and f

$\Longleftarrow$ f is dif

$\not\Longrightarrow$

$\Longleftrightarrow$ $f(x\ y)$ c

by $y = f(x_0, y_0)$

near $(x_0, y_0, f(x$

$\Longleftrightarrow$ $f(x, y) - L$

$= o\left(\sqrt{(x - x_0)^2 +}\right.$

$\Longleftrightarrow$ Existence of tan

$f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist

$\Longleftarrow$ f is diff. at (x_0,y_0)

$\not\Longrightarrow$

$\Longleftrightarrow$ $f(x,y)$ can be appx.

by $\cancel{y}\ L(x,y) = \underbrace{f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}_{L(x,y)}$

near $(x_0,y_0,f(x_0,y_0))$

$$\Longleftrightarrow f(x,y) - L(x,y) = o\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right)$$

$$\left(\begin{array}{l}\text{or} \\ =\varepsilon_1(x-x_0) \\ +\varepsilon_2(y-y_0) \\ \left(\lim \varepsilon_1,\varepsilon_2 = 0\right.\end{array}\right)$$

$\Longleftrightarrow$ Existence of tangent plane

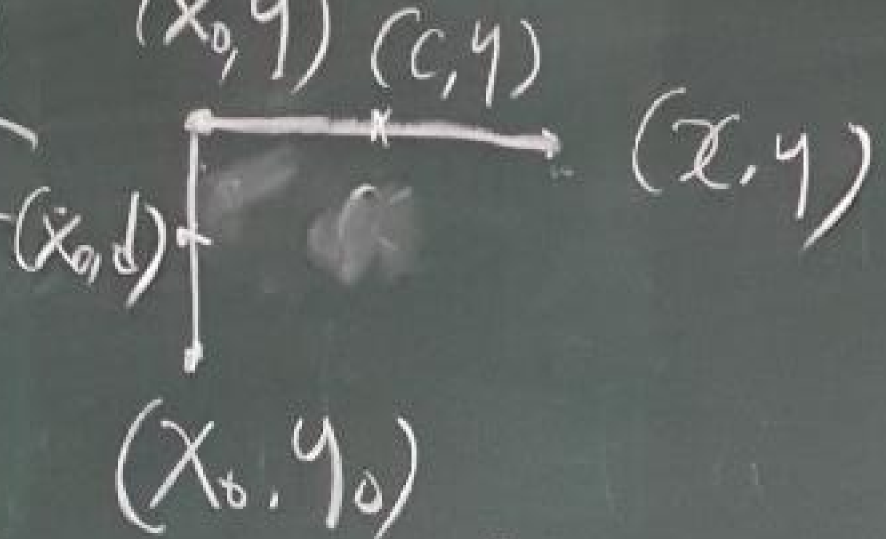
Thm R: an open region

$(x_0, y_0) \in R$. If f is defined on R and f_x, f_y are cont. at (x_0, y_0) then f is diff. at (x_0, y_0)

Def R is open if for any $(x_0, y_0) \in R$, there exists a corresponding $\delta > 0$, such that

$$\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta \Rightarrow (x,y) \in R$$

pf



$$f(x,y)-f(x_0,y_0)$$
$$=\big(f(x,y)-f(x_0,y)\big)+\big(f(x_0,y)-f(x_0,y_0)\big)$$
$$=f_x(c,y)(x-x_0)+f_y(x_0,d)(y-y_0)$$

c between x and x_0, d between y and y_0

$$=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$
$$+\underbrace{\big(f_x(c,y)-f_x(x_0,y_0)\big)}_{\varepsilon_1}(x-x_0)+\underbrace{\big(f_y(x_0,d)-f_y(x_0,y_0)\big)}_{\varepsilon_2}(y-y_0)$$

Thm

pf

Thm If f is diff. at (x_0, y_0)
then f is cont. at (x_0, y_0)

pf diff.:

$$\Leftrightarrow f(x,y)-f(x_0,y_0) = f_x(x-x_0) + f_y(y-y_0) + \underbrace{\boxed{\varepsilon_1(x-x_0) + \varepsilon_2(y-y_0)}}_{= o(x-x_0)+o(y-y_0)}$$

Cont.

$$\Leftrightarrow f(x,y)-f(x_0,y_0) = \varepsilon \left(= o(1)\right)$$

$$\lim_{(x,y)\to(x_0,y_0)} \varepsilon_1, \varepsilon_2, \varepsilon = 0$$

$\therefore$ diff. $\Rightarrow$ cont.

Thm If $f, f_x, f_y, f_{xy}, f_{yx}$

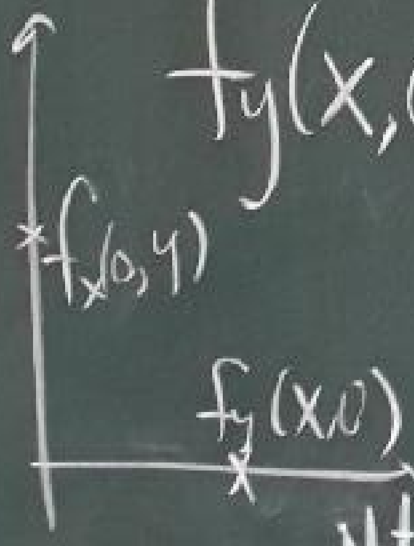
are all cont. at $(x_0, y_0) \in R$ (open)

Then $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$.

Eg: $f(x,y) = \begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

pf $f_x(0,y) = y\left(\dfrac{x^2-y^2}{x^2+y^2}\right)\Big|_{x=0} + xy\left(\dfrac{x^2-y^2}{x^2+y^2}\right)\Big|_{x=0} = -y$

$f_y(x,0) = x\left(\dfrac{x^2-y^2}{x^2+y^2}\right)\Big|_{y=0} + xy\left(\dfrac{x^2-y^2}{x^2+y^2}\right)\Big|_{y=0} = x$



$f_{xy}(0,0) = -1 \neq f_{yx}(0,0) = 1$

Note: $f_x(0,0) = 0 = f_y(0,0)$

The Chain Rule

Single variable:

$$\frac{d}{dx} f(g(x)) = \frac{df}{dy}\Big|_{y=g(x)} \cdot \frac{dg(x)}{dx}$$

Multi-variable (assume f is diff)

$$\frac{d}{dt} f(x(t), y(t))\Big|_{t=t_0} = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{f(x(t+\Delta t), y(t+\Delta t)) - f(x(t), y(t))}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \begin{pmatrix} f_x(x(t), y(t)) \\ + \varepsilon_1 \end{pmatrix} \left(\frac{x(t+\Delta t) - x(t)}{\Delta t} \right) + \begin{pmatrix} f_y(x(t), y(t)) \\ + \varepsilon_2 \end{pmatrix} \left(\frac{y(t+\Delta t) - y(t)}{\Delta t} \right)$$

$$= f_x(\cdot)\, x'(t) + f_y(\cdot\cdot)\, y'(t)$$

Similarly,

$$\frac{\partial}{\partial s} f(x(s,t), y(s,t), z(s,t))$$

$$= f_x(\cdot) x_s(s,t) + f_y(\cdot) y_s(s,t) + f_z(\cdot) z_s(s,t)$$

and similarly for

$$\frac{\partial}{\partial t} f(x(s,t), y(s,t), z(s,t))$$