Eg $$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$$

Remark: domain $= \left\{ \begin{array}{l} x \geq 0,\ y \geq 0 \\ x \neq y \end{array} \right\}$

Note: $(0,0) \notin$ domain

$$f(x,y) = \frac{x(x-y)}{\sqrt{x}-\sqrt{y}} = x(\sqrt{x}+\sqrt{y})$$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = 0(\sqrt{0}+\sqrt{0}) = 0$$

Eg $\lim_{(x,y)\to(0,0)} \frac{y}{x} = ?$

Let $\begin{cases} x = r\cos\theta,\ y = r\sin\theta & \text{— (1)} \\ \text{or } y = mx. & \text{— (2)} \end{cases}$

$$f(x,y) = \begin{cases} \tan\theta & \text{— (1)} \\ m & (2) \end{cases}$$

different θ, or different $m \in \mathbb{R}$ gives different limit

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

<u>Thm</u> (Two path Theorem)

If $f(x,y)$ has different limits along two paths passing through (x_0, y_0) as $(x,y) \to (x_0, y_0)$

Then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist!

Eg $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$

Let $\begin{cases} x = r\cos\theta,\ y = r\sin\theta \\ \text{or } y = mx \end{cases}$

$\Rightarrow f(x,y) = \begin{cases} 2\cos\theta\sin\theta \\ \text{or } \dfrac{2m}{1+m^2} \end{cases}$

Different θ (or m) gives different answer.

Two path Thm $\Rightarrow$ lim does not exist

Eg $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2y}{x^4+y^2}$

(i) Let $\begin{cases} x = r\cos\theta,\ y = r\sin\theta \\ \text{or } y = mx \end{cases}$

$$\lim_{x\to 0} \frac{2x^2 mx}{x^4+m^2x^2} = 0$$

$$\lim_{r\to 0} \frac{2r^3\cos^2\theta\sin\theta}{r^4\cos^4\theta + r^2\sin^2\theta} = 0$$

In other words

$f(x,y) \to 0$ when

$(x,y) \to (0,0)$ along straight lines.

But $f(x,y) \to \dfrac{2k}{1+k^2}$

when $(x,y) \to (0,0)$ along $y = kx^2$

From two path theorem,

the limit does not exist.

Continuity of $f(x,y)$

<u>Def</u> $f(x,y)$ is cont. at (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

i.e. Given $\varepsilon > 0$, there exists a corresponding $\delta > 0$, such that

$$\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta \Rightarrow |f(x,y)-f(x_0,y_0)| < \varepsilon$$

Eg $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Then (i) $f(x,y)$ is cont. at $(x,y) \neq (0,0)$

(ii) $f(x,y)$ is not cont at $(0,0)$

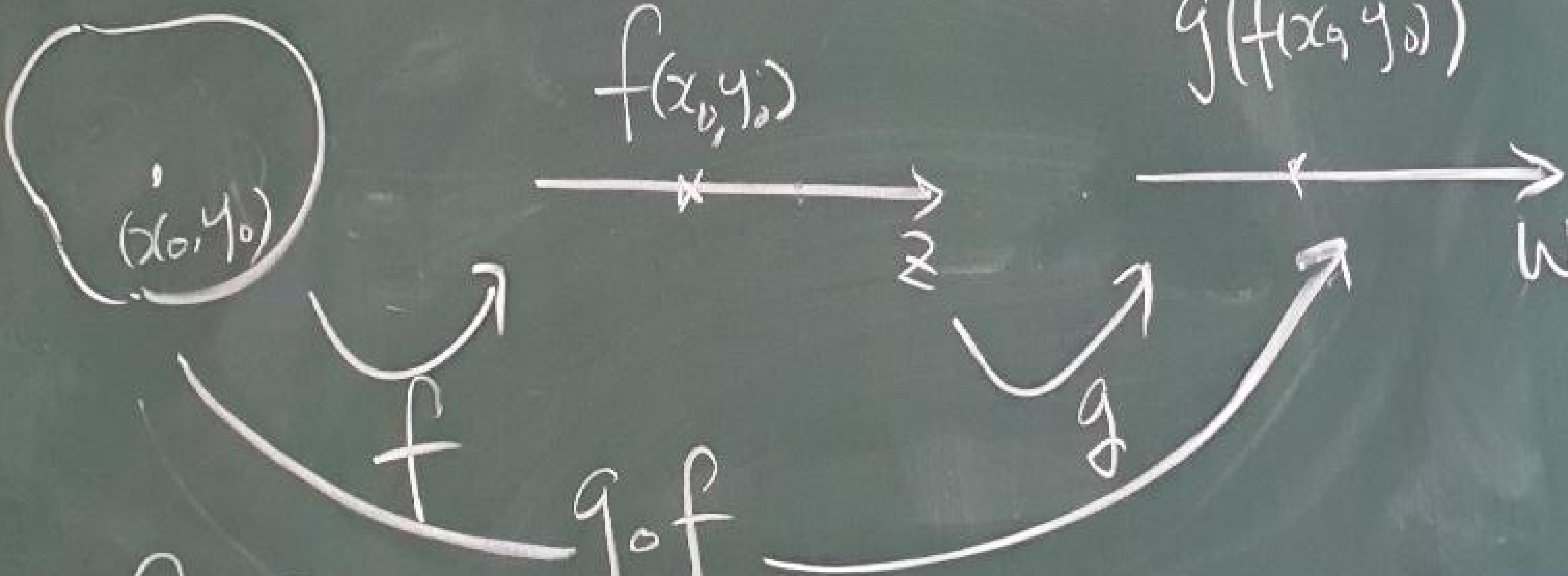
Eg: $f(x,y) = \begin{cases} \dfrac{2x^2y}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Then: Same conclusion as above.

Eg $w = \cos\left(\dfrac{xy}{x^2+1}\right)$

is continuous at any (x, y)

$$(x,y) \xrightarrow{\ f\ } z = \frac{xy}{x^2+1} \xrightarrow{\ g\ } w = \cos z$$



If $\begin{cases} f \text{ is cont at } (x_0, y_0) \\ g \text{ is cont at } f(x_0, y_0) \end{cases}$

Then $g \circ f$ is cont at (x_0, y_0)

Partial Derivatives

Def $\dfrac{\partial f}{\partial x}(x_0, y_0) = \lim\limits_{h \to 0} \dfrac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$

$\dfrac{\partial f}{\partial y}(x_0, y_0) = \lim\limits_{h \to 0} \dfrac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$

Notation: $\dfrac{\partial f}{\partial x} = f_x = \partial_x f$

$\dfrac{\partial f}{\partial y} = f_y = \partial_y f$

Eg $f(x,y) = x^2 + 3xy + y - 1$

$$\partial_x f(x,y) = 2x + 3y + 0$$

$$\partial_y f(x,y) = 0 + 3x + 1$$

Eg: Find $\frac{\partial z}{\partial x}$ if

$$yz - \ln z = x + y$$

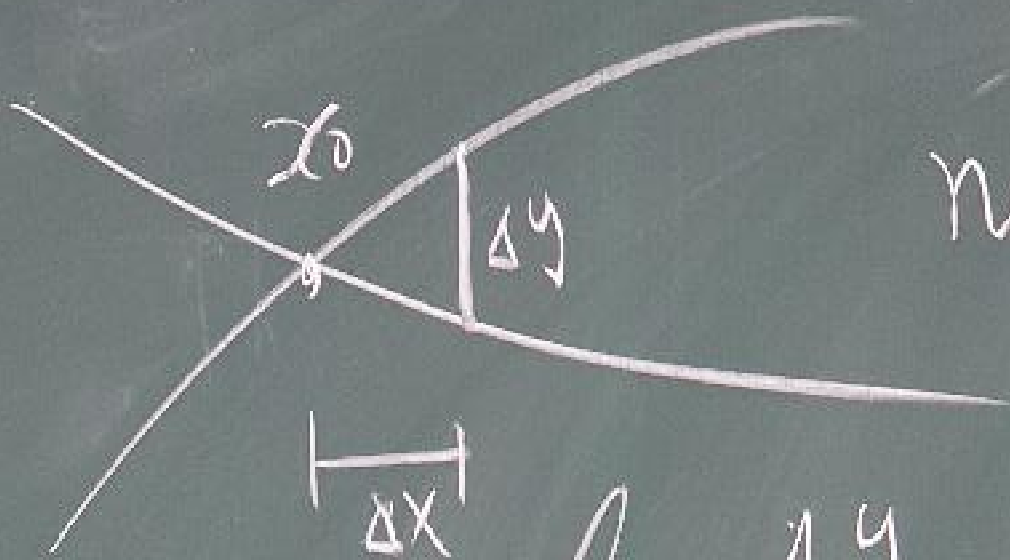
$$(y \cdot z(x,y) - \ln z(x,y) = x + y)$$

$$y z_x - z_x / z = 1$$

$$z_x = \frac{1}{y - \frac{1}{z}}$$

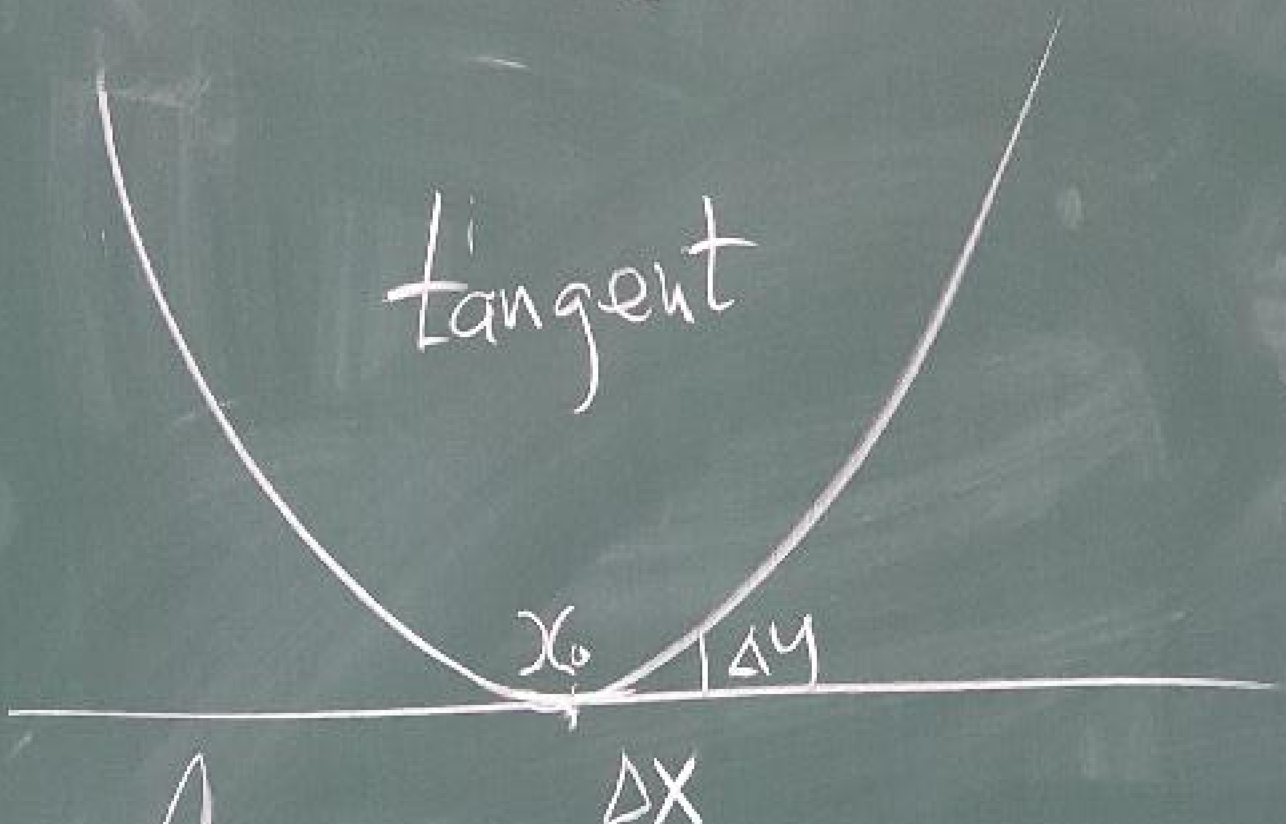
Def: $z = f(x,y)$ is differentiable at (x_0, y_0) if it has a tangent plane $z = L(x,y)$
$= a(x-x_0) + b(y-y_0) + c$
at $(x_0, y_0, f(x_0, y_0))$

1D tangent Vs not tangent



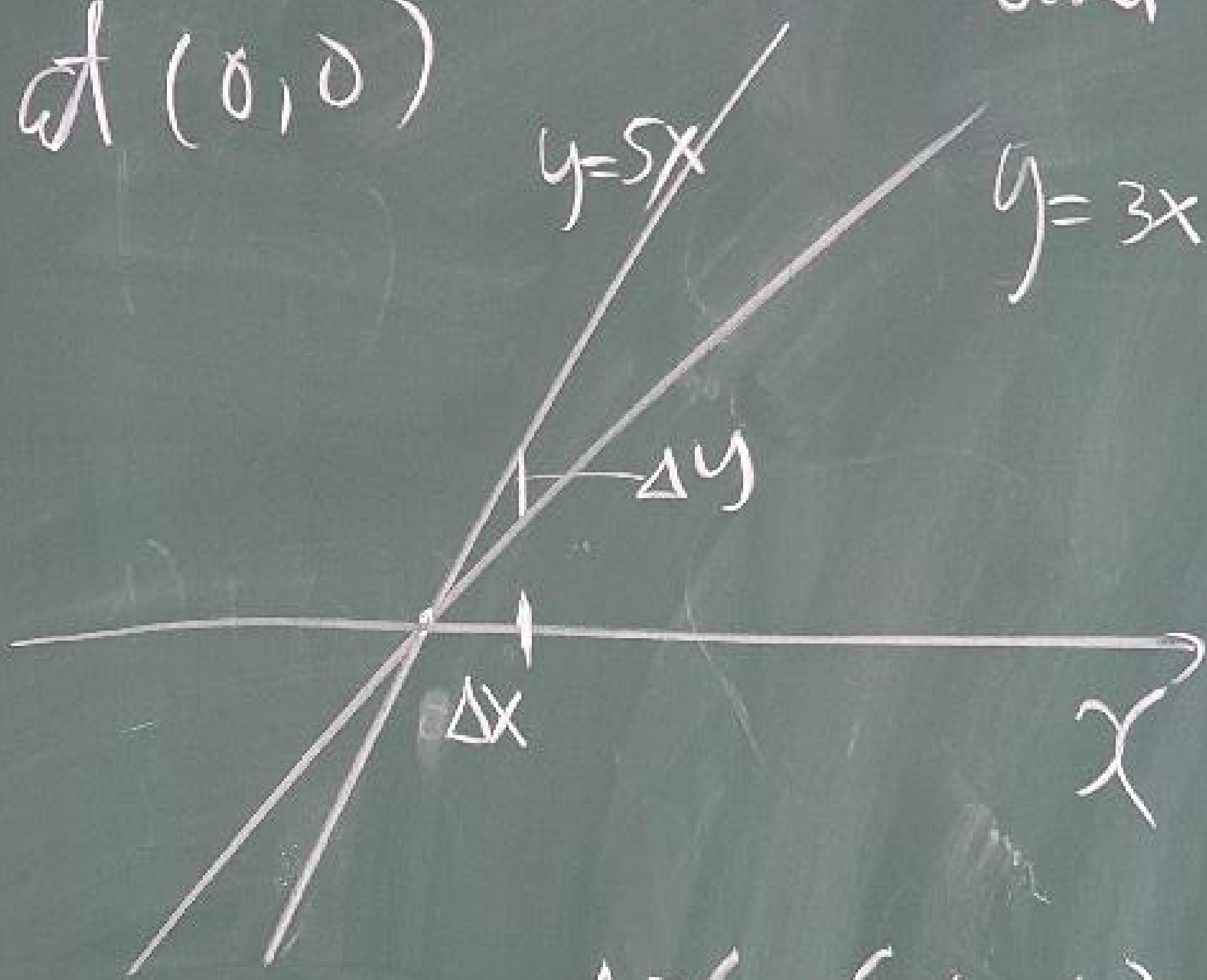
not tangent

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \neq 0$$



$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0$$

Eg $L_1: y = 3x$ and $L_2: y = 5x$ are not tangent to each other at $(0,0)$



$x_0 = 0, \quad \Delta x = (x - x_0) = x$

$\Delta y = 5x - 3x = 2x$

$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2.$

Eg: $f(x) = x^2$, $x_0 = 0$

"Slope of tangent at x_0" $= f'(x_0) = 0$

tangent line $y = 0$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x \to 0} \frac{x^2}{x} = 0$$

$$\Delta x = x - x_0 = x - 0 = x$$

In general
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{f(x) - (f(x_0) + f'(x_0)(x - x_0))}{x - x_0}$$
$$= 0$$

Def $z = f(x,y)$

and $z = g(x,y)$

are tangent at (x_0, y_0, z_0)

if $\lim\limits_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y) - g(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$

$$\left(\lim_{\substack{(\Delta x, \Delta y) \\ \to (0,0)}} \frac{\Delta z}{\sqrt{\Delta x^2 + \Delta y^2}} = 0 \right)$$