Eg: $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=?$

Sol $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\cdots \quad |x|<1$

If $0<x\leq 1$

$$|R_n(x)|=\left|\frac{n!\,(1+c_{n+1})^{-n-1}}{(n+1)!}\,x^{n+1}\right|\xrightarrow[n\to\infty]{}0$$

$$0<c_{n+1}<x\leq 1$$

$$\Longrightarrow x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\cdots=\ln(1+x),\ 0<x\leq 1$$

~~What about $x=1$?~~

Another way to find the sum of the series

Recall

$t \neq -1 \Rightarrow \frac{1}{1+t} = 1 - t + t^2 + \cdots + (-1)^{n-1} t^{n-1} + \frac{(-1)^n t^n}{1+t}$

$\int_0^x dt \Rightarrow \ln(1+x) = x - \frac{x^2}{2} + \cdots (-1)^{n-1} \frac{x^n}{n} + \tilde{R}_n(x)$

$$\tilde{R}_n(x) = \int_0^x \frac{(-1)^n t^n}{1+t} dt, \ -1 < x$$

$$\left|\tilde{R}_n(x)\right| = \int_0^x \frac{t^n}{1+t} dt \leq \begin{cases} \int_0^x t^n dt = \frac{x^{n+1}}{n+1}, & \underline{0 < x} \\ \frac{1}{1+x} \int_0^x t^n dt = \frac{x^{n+1}}{(1+x)(n+1)}, & -1 < x < 0 \end{cases}$$

$\therefore \lim_{n \to \infty} \tilde{R}_n(1) = 0. \ \therefore \text{Ans} = \ln 2$

See supplement for more details

In fact, (beyond this course)

$$\frac{(-1)^n t^n}{1+t} \xrightarrow{n\to\infty} 0 \text{ uniformly on } [0,x]$$

(uniform convergence)

$$\Rightarrow \int_0^x \frac{(-1)^n t^n}{1+t} \xrightarrow[n\to\infty]{} 0$$

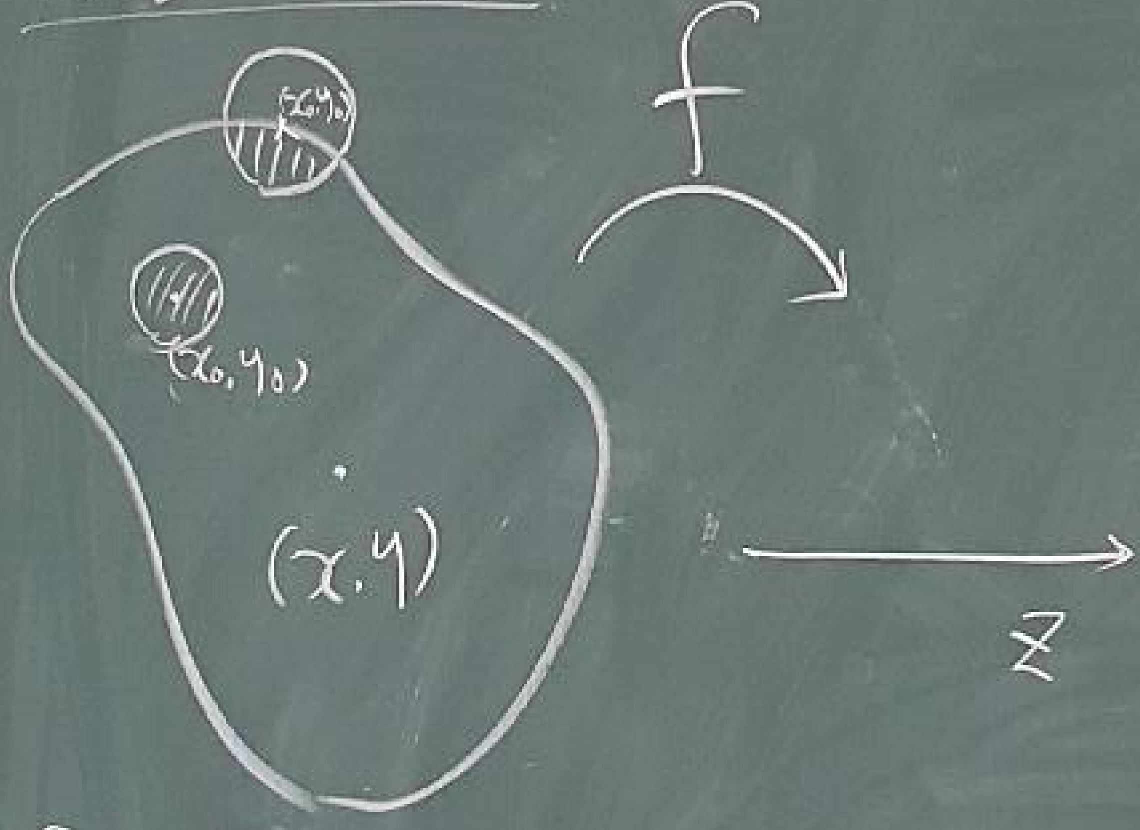
Similarly

$$\frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5}, \cdots \quad |x|<1$$

Limit and Continuity in higher dimensions.

2D case



$$f : D_f \longrightarrow \mathbb{R}$$

(domain of f)

$$(x, y) \qquad z = f(x, y)$$

Def $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$

if for any $\varepsilon > 0$,
there exists a corresponding $\delta > 0$
such that

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \Rightarrow |f(x,y) - L| < \varepsilon$$

(and $(x,y) \in D_f$)

<u>Eg</u> $\lim\limits_{(x,y)\to(x_0,y_0)} x = x_0$

<u>Pf</u> Take $\delta = \varepsilon$

$0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \varepsilon$

$f(x,y) = x \begin{array}{l} < x_0+\varepsilon \\ > x_0-\varepsilon \end{array}$

$\Rightarrow |f(x,y) - x_0| < \varepsilon$

Eg $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$

Let $x = r\cos\theta$, $y = r\sin\theta$

"$0 < \sqrt{(x-0)^2+(y-0)^2} < \delta \iff 0 < r < \delta$

$\dfrac{4xy^2}{x^2+y^2} = 4r\cos\theta\sin^2\theta$

$|f(x,y) - 0| < 4\delta$ $\left(\begin{array}{l}\text{Given } \varepsilon > 0 \\ \text{take } \delta = \dfrac{\varepsilon}{4}\end{array}\right)$