Binomial Series

$T_{(1+x)^m,0}(x) = ?$ $m \in \mathbb{R}$

(i) $m \in \mathbb{N}$. $T_{(1+x)^m,0}(x) = (1+x)^m$
$x \in \mathbb{R}$

(ii) $m \notin \mathbb{N}$, let $f(x) = (1+x)^m$

$$f'(x) = m(1+x)^{m-1}$$

$$f''(x) = m(m-1)(1+x)^{m-2}$$

$$(1+x)^m = P_n(x) + R_n(x)$$

$$P_n(x) = \sum_{k=0}^{n} \binom{m}{k} x^k$$

where $\binom{m}{k} = \dfrac{m(m-1)\cdots(m-k+1)}{k!}$

From Taylor's formula.

$$R_n(x) = \binom{m}{n+1}(1+c_{n+1})^{m-n-1} x^{n+1}$$

c_{n+1} between 0 and x.

Firstly, one can show that

$$T_{(1+x)^m,0}(x)=\sum_{k=0}^{\infty}\binom{m}{k}x^k$$

conv. (abs.) on $|x|<1$ $\left(\begin{matrix}\text{Ratio}\\ \text{Test}\end{matrix}\right)$

Q: $\lim_{n\to\infty} R_n(x) \overset{?}{=} 0$ on $|x|<1$

Ans: if $0<x<1$ and $n+1>m$

$$\Rightarrow 0<(1+c_{n+1})^{m-n-1}<1$$ (Yes, if $0<x<1$)

But not clear if $-1<x<0$

since $(1+c_{n+1})^{m-n-1}>1$.

It can be shown (homework) that indeed $\lim_{n\to\infty} R_n(x) = 0$. (*)

on $|x|<1$, $\forall m \in \mathbb{R}$.

Eg: $\sqrt{1+x} = (1+x)^{\frac{1}{2}}$

$$= 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{3!}x^3 + \cdots$$

on $|x|<1$

$\underline{\text{Eg}}$ $T_{\sin^{-1}x,0}(x) = ?$

$\underline{\text{Sol}}$: $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} = (1+(-x^2))^{\frac{-1}{2}}$

$$= 1 + \left(\frac{-1}{2}\right)(-x^2) + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2!}(-x^2)^2 + \cdots$$

$$\Rightarrow \int_0^x \frac{d}{dt}\sin^{-1}t\,dt = \int_0^x \left(1 + \frac{-1}{2}(-t^2) + \cdots\right)dt$$

$$\therefore T_{\sin^{-1},0}(x) = x + \frac{x^3}{2\cdot 3} + \frac{3}{5\cdot 2^3}x^5 + \cdots$$

From $(*)$ $\Rightarrow$ $\sin^{-1}x = T_{\sin^{-1},0}(x)$ on $|x|<1$

$\underline{\text{Eg}}$ $T_{\tan x, 0}(x) = ?$ $\begin{pmatrix}\text{up to} \\ x^5\end{pmatrix}$

If $\tan x = \sum_{k=0}^{\infty} C_k x^k$ on $|x|<R$, $R>0$.

then $T_{\tan x, 0}(x) = \sum_{k=0}^{\infty} C_k x^k$

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)(1-y)^{-1}, \quad y = \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots$$

$$\doteq \left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)\left(1+y+y^2+\cdots\right)$$

Note $y = 1-\cos x$

$$\Rightarrow |y|<1 \text{ on } |x|<\frac{\pi}{2}$$

$$\therefore \tan x = \left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)$$

$$\left(1+\left(\frac{x^2}{2!}-\frac{x^4}{4!}+\cdots\right)+\left(\frac{x^2}{2!}-\cdots\right)^2+\cdots\right)$$

$$= x\left(1-\frac{x^2}{6}+\frac{x^4}{120}-\cdots\right)\left(1+\frac{x^2}{2}+\frac{5}{24}x^4+\cdots\right)$$

$$= x+\frac{1}{3}x^3+\frac{2}{15}x^5+\cdots\cdots$$

<u>Note</u>: The above shows that (i) Radius of conv for $T_{\tan,0}(x)$ is at least $\frac{\pi}{2}$ (may be larger).

(ii) $T_{\tan,0}(x) = \tan x$ on $|x| < \frac{\pi}{2}$

For this problem, we only ~~care~~ ask about what C_k's are

(Radius of conv or whether $T_{\tan x,0}(x) = \tan(x)$ on $|x| < R$ are separate questions)

Eg. $\displaystyle\lim_{x \to 0} \frac{\sin x - \tan x}{x^3}$.

Firstly, we can compute

$\tan x = \dfrac{\sin x}{\cos x}$ (yet another method)

$$
\begin{array}{r|llll}
 & 0 & 1 & 0 & \frac{1}{3} \\
\hline
(1 \;\; 0 \;\; \frac{-1}{2}, \dots\dots) & 0 & 1 & 0 & \frac{-1}{6} \cdots \\
 & & 1 & 0 & \frac{-1}{2} \\
\hline
 & & 0 & 0 & \frac{1}{3}
\end{array}
$$

(long divison)

$= x + \frac{1}{3}x^3 + \cdots$

$$\therefore \lim_{x \to 0} \frac{\sin x - \tan x}{x^3}$$

$$= \lim_{x \to 0} \frac{\left(\left(x - \frac{x^3}{6} + \cdots\right) - \left(x + \frac{1}{3}x^3 + \cdots\right) \right)}{x^3}$$

$$= -\frac{1}{2}$$

Eg $\lim_{x\to 0} \frac{1}{x}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$

$$=\lim_{x\to 0} \frac{x-\sin x}{x^2\sin x}$$

$$=\lim_{x\to 0} \frac{x-\left(x-\frac{x^3}{3!}+\cdots\right)}{x^3-\cdots\cdots}$$

$$=\frac{1}{6}$$

Eg: $\frac{1}{1\cdot 2^1} - \frac{1}{2 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \cdots$

Sol: $f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ conv on $|x|<1$
(ratio or root)

$f(\frac{1}{2}) = ?$

$$f'(x) = \sum_{n=1}^{\infty} (-1)^{n-1} x^{n-1} = \sum_{m=0}^{\infty} (-1)^m x^m$$

$$= \frac{1}{1+x} \quad \text{on } |x|<1$$

$$\therefore f(x) = f(0) + \int_0^x f(t)\,dt = 0 + \int_0^x \frac{1}{1+t}\,dt$$

$$= \ln(1+x) \quad \therefore f(\tfrac{1}{2}) = \ln(\tfrac{3}{2})$$

Remark. Formally, we can write

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i\sin\theta.$$

Eg: $\int e^{ax} \cos bx \, dx = ?$

$a, b \in \mathbb{R}$

$$= \int e^{ax} \operatorname{Re}\left(e^{ibx}\right) dx$$

$$= \operatorname{Re} \int e^{ax} e^{ibx} \, dx$$

$$= \operatorname{Re} \int e^{a+ibx} \, dx$$

$$= \operatorname{Re}\left(\frac{e^{(a+ib)x}}{a+ib} \cdot \frac{a-ib}{a-ib} \right) + C$$

$$= \mathrm{Re}\left(\frac{e^{ax}(\cos bx + i\sin bx)(a - ib)}{a^2+b^2} \right) + C$$

$$= \frac{e^{ax}\left(a\cos(bx) + b\sin(bx)\right)}{a^2+b^2} + C \quad (\bigstar)$$

$C_b = \cos(bx)$

$S_b = \sin(bx)$

<u>Check</u> $\frac{d}{dx}(\bigstar)$

$$= \frac{e^{ax}}{a^2+b^2}\left(a^2 C_b + ab S_b - ab S_b + b^2 C_b\right)$$

$$= e^{ax} C_b = e^{ax}\cos bx \quad \therefore \text{correct!}$$