Taylor's Thm (Taylor's formula)

If $f, f', \ldots, f^{(n)}, \ldots$

all exist on $(a-R, a+R)$

Then for any $n \in \mathbb{N}$, $|x-a|<R$

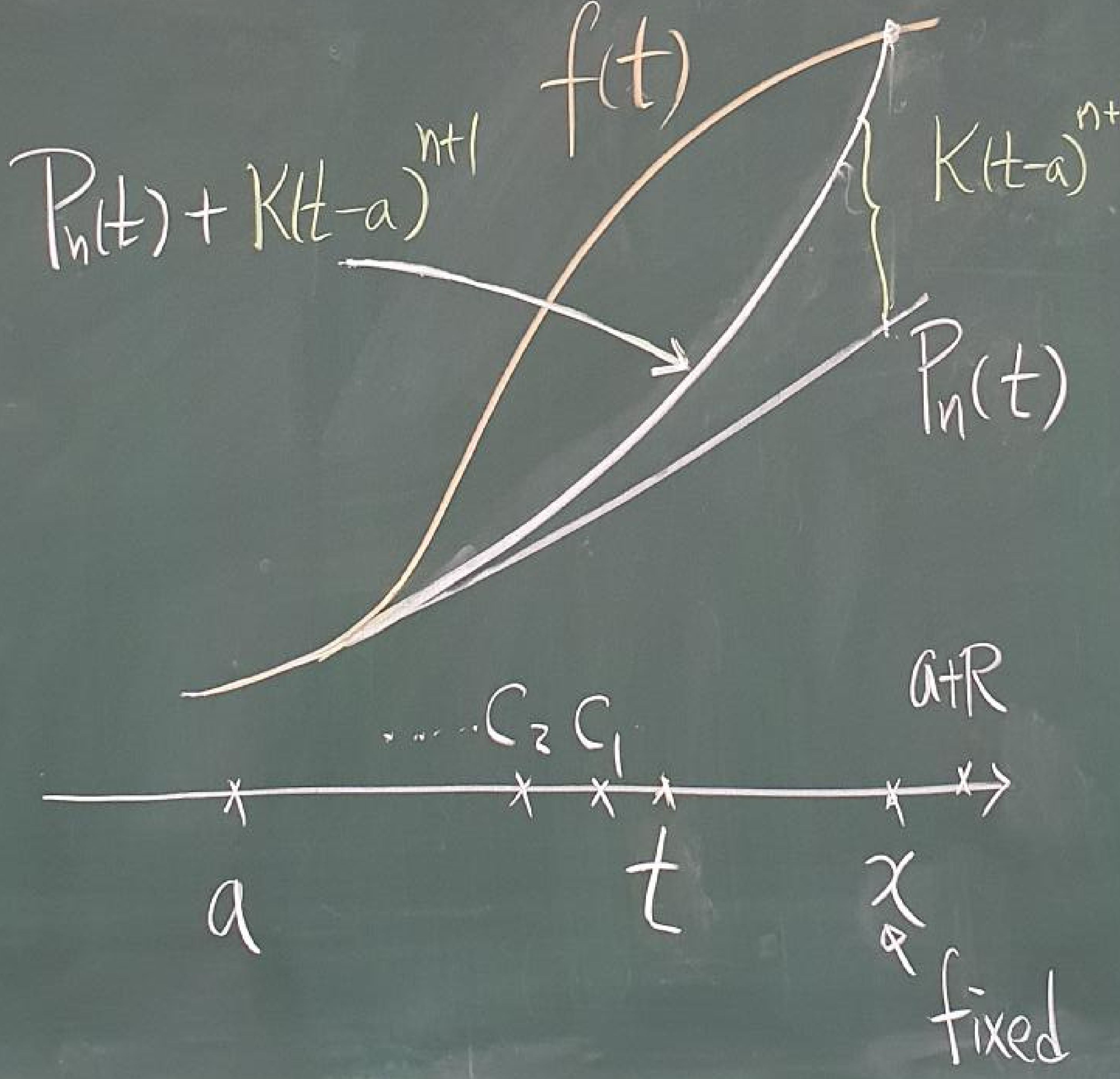
$$f(x) = P_n(x) + R_n(x)$$

where $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$

$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, c between a & x

Pf Suppose $|x-a|<R$.

Assume $x>a$. ($x<a$ similar)



$$P_n(t) = f(a) + f'(a)(t-a) + \dots + \frac{f^{(n)}(a)}{n!}(t-a)^n$$

First, we find $K \in \mathbb{R}$ such that

$$f(x) = P_n(x) + K(x-a)^{n+1},$$

$$\Rightarrow K = \frac{f(x) - P_n(x)}{(x-a)^{n+1}}$$

Then consider

$$F(t) = f(t) - \left(P_n(t) + K(t-a)^{n+1}\right)$$

$$F(a) = 0 = F(x) \Longrightarrow F'(c_1) = 0,\ c_1 \in (a, x)$$

$(c_1 \in (x, a)$ if $x < a)$

$$F'(a) = f'(a) - \left(P_n'(a) + (n+1)K(a-a)^n\right) = 0$$

$$F'(a) = 0 = F'(c_1) \Longrightarrow F''(c_2) = 0,\ c_2 \in (a, c_1)$$

$$F''(a) = f''(a) - \left(P_n''(a) + K(n+1)n(a-a)^{n-1}\right)$$

$$F''(a) = 0 = F''(c_2) \Rightarrow F'''(c_3) = 0,\ c_3 \in (a, c_2)$$

$$\vdots$$

$$\Rightarrow F^{(n)}(c_n) = 0,\quad c_n \in (a, c_{n-1})$$

$$F^{(n)}(a) = 0 = F^{(n)}(c_n) \Rightarrow F^{(n+1)}(c) = 0,\ c \in (a, c_n)$$

$$0 = F^{(n+1)}(c) = f^{(n+1)}(c) - K(n+1)!$$

$$\therefore \frac{f(x) - P_n(x)}{(x-a)^{n+1}} = K = \frac{f^{(n+1)}(c)}{(n+1)!}$$

$$\Rightarrow f(x) = P_n(x) + R_n(x)$$

<u>Cor</u> $T_{f,a}(x) = f(x)$

$$\Leftrightarrow \lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

$$= 0$$

Thm If $\exists\, M > 0$ such that $|f^{(n+1)}(t)| \leq M$ for all t between a and x and all $n \in \mathbb{N}$

Then $$|R_n(x)| \leq \frac{M\,|x-a|^{n+1}}{(n+1)!} \xrightarrow{n \to \infty} 0,$$ and $T_{f,a}(x) = f(x)$

$\underline{\text{Eg}}$. $\overline{T}_{\sin,a}(x) = \sin x \; \forall x \in \mathbb{R}$

$\because \left| f^{(n+1)}(c) \right| = \left| \pm \begin{matrix} \cos c \\ \sin c \end{matrix} \right| \leq 1$

Similarly, $\overline{T}_{\cos,a}(x) = \cos x$

$\underline{\text{Eg}}$ $\overline{T}_{e^x,a} = e^x$

$\because \left| f^{(n+1)}(c) \right| = e^c \begin{cases} \leq e^x, & a < x \\ \leq e^a, & x < a \end{cases}$

$\leq \text{Max}(e^x, e^a) \equiv M$

$\therefore \overline{T}_{e^x,a}(x) = e^x, \; \forall x \in \mathbb{R}$.

Known Power Series Representations

* $(1\pm x)^{-1}$, $|x|<1$, geometric series

* $\tan^{-1}x$, $|x|<1$
  $\ln(1\pm x)$, $|x|<1$ (geometric series + term by term integration)

* $P(x)$: Polynomial

* $T_{e^x,a} = e^x$, $x \in \mathbb{R}$

* $T_{\sin,a}(x) = \sin x$. In particular $(a=0)$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\cdots$