Taylor Series

Thm. If $f(x)=\sum_{k=0}^{\infty} a_k (x-a)^k$

converges on $|x-a|<R$, $R>0$

Then $a_n = \dfrac{f^{(n)}(a)}{n!}$

pf: $f^{(n)}(a) = \dfrac{d^n}{dx^n}\left(a_n(x-a)^n + \sum_{k\neq n}\right)\Big|_{x=a}$

$= n!\, a_n$

Note:

$$\boxed{f(x)=\sum_{k=0}^{\infty} a_n(x-a)^n \Rightarrow a_n=\frac{f^{(n)}(a)}{n!}}$$

$$\not\Longrightarrow f(x)=\underbrace{\sum_{k=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n}_{(*)}$$

(except for $x=a$)

Note: Let R = radius of conv. for $(*)$

R may or may not >0

Even if $R>0$, $(*)$ need not $\underset{(x\neq a)}{=} f(x)$

Eg. $f(x)=\begin{cases} 0 & x=0 \\ e^{\frac{-1}{x^2}}, & x\neq 0 \end{cases}$

$f(0)=0$

$f'(0)=\lim_{h\to 0}\frac{e^{\frac{-1}{h^2}}-0}{h-0}\overset{l'Hôpital}{=\!=}0$

$f'(x)_{x\neq 0}=2x^{-3}e^{-x^{-2}}$

$f''(0)=\lim_{h\to 0}\frac{f'(h)-f'(0)}{h-0}=\lim_{h\to 0}\frac{2e^{-h^{-2}}}{h^4}$

$=\lim_{h\to 0}\frac{2h^{-4}}{e^{h^{-2}}}=\lim_{h\to 0}\frac{-8h^{-5}}{-2h^{-3}e^{h^{-2}}}=\lim_{h\to 0}\frac{4h^{-2}}{e^{h^{-2}}}=0$

Similarly

$$f''(0)=0,\ f'''(0)=0,\ \cdots\cdots$$

$$f(x) \overset{?}{=} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k, \quad x \neq 0.$$

$\vee$ No! $\parallel$

0 $\underline{\underline{}}$ 0

This example shows

$$f(x) \neq \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

for any $x \neq 0$

$\underline{\text{Def}}$ $T_{f,a}(x)$ (Taylor series generated by f centered at a)

$$\stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$0! \stackrel{\text{def}}{=} 1$$

Special case: $a=0$

also called McLaurin Series (generated by f)

<u>Eg</u> Let $P(x)$ be a polynomial

$$P(x)=\sum_{k=0}^{m} b_k x^k$$

$$(*)\quad T_{P,0}(x)=\sum_{k=0}^{\infty}\frac{\frac{d^kP(0)}{dx^k}}{k!}x^k$$

$$\frac{d^jP(0)}{dx^j}=\begin{cases} j!\,b_j, & 0\le j\le m \\ 0, & j>m\end{cases}$$

Substitute back to $(*)$

$$\Rightarrow T_{P,0}(x)=P(x)\left(=T_{P,a}(x)\ \ \forall a\in\mathbb{R}\right)$$

Eg: $T_{e^x,a}(x) = ?$

Sol: $\left.\frac{d^k e^x}{dx^k}\right|_{x=a} = e^a$

$\therefore T_{e^x,a}(x) = e^a \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!}$

*: Radius of conv: $R = \infty$

* In fact $T_{e^x,a}(x) = e^x$ for any $x \in \mathbb{R}$ (Next Section)

Eg: $T_{\cos x, 0}(x) = ?$

Sol Let $f(x) = \cos x$

$$\Rightarrow f(0),\ f'(0),\ f''(0),\ f'''(0),\ f^{(4)}(0),\ \cdots$$

$$\cos 0,\ -\sin 0,\ -\cos 0,\ \sin 0,\ \cos 0,\ \cdots$$

$$1,\ 0,\ -1,\ 0,\ 1,\ \cdots$$

$$\therefore T_{\cos x, 0}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Ratio test $\Rightarrow R = \infty$

(also next time: $T_{\cos x, 0}(x) = \cos x$ for any $x \in \mathbb{R}$)

Eg $\int_0^{\frac{1}{2}} \sin t^2 \, dt = ?$

Sol: $T_{\sin x, 0}(x) = \sin x \quad \forall x \in \mathbb{R}$

(Next time) $\quad (R = \infty)$

$$\parallel$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad (\text{Exercise})$$

$$\therefore \int_0^{\frac{1}{2}} \sin t^2 \, dt = \int_0^{\frac{1}{2}} \left(t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \cdots \right) dt$$

$$= \left. \frac{t^3}{3} - \frac{t^7}{7 \cdot 3!} + \frac{t^{11}}{11 \cdot 5!} \right|_0^{\frac{1}{2}} + \text{error}$$

$$|\text{error}| < \frac{(\frac{1}{2})^{15}}{15 \cdot 7!} \leq \frac{1}{1024} \frac{1}{32 \cdot 15} \frac{1}{5040} < 10^{-9}$$

Taylor's formula

If $f, f', f'', \ldots, f^{(n)}, \ldots$

all exist on $|x-a|<R$, $R>0$

Then for any $n \in \mathbb{N}$, $|x-a|<R$

We have $f(x) = P_n(x) + R_n(x)$

where $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x