Term by term differentiation

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} C_n (x-a)^n\right) = ?$$

Thm If $f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$

Converges (abs.) on $|x-a| < R$

Then ①. $f', f'', \cdots f^{(n)}$ all

exist on $|x-a| < R$

② $f'(x) = \sum_{n=1}^{\infty} n C_n (x-a)^{n-1}$,

$f''(x) = \sum_{n=2}^{\infty} n(n-1) C_n (x-a)^{n-2}$

etc. ...... on $|x-a| < R$

Eg $\sum_{n=0}^{\infty}(n+1)\left(\frac{1}{2}\right)^n$

$= 1 + 2\left(\frac{1}{2}\right) + 3\cdot\left(\frac{1}{2}\right)^2 + \cdots = ?$

Sol $= \sum_{n=0}^{\infty}(n+1)x^n \Big|_{x=\frac{1}{2}}$

$= \sum_{m=1}^{\infty} m\, x^{m-1} \Big|_{x=\frac{1}{2}}$

$= \frac{d}{dx}\left(\sum_{m=0}^{\infty} x^m\right) \Big|_{x=\frac{1}{2}}$ $\underbrace{|x|<1}$ conv

$= \frac{d}{dx}\left(\frac{1}{1-x}\right)\Big|_{x=\frac{1}{2}} = \frac{1}{(1-x)^2}\Big|_{x=\frac{1}{2}} = 4$

Remark: $\sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$

conv. (abs.) for any $x \in \mathbb{R}$.

but "term by term $\frac{d}{dx}$"

$$\sum_{n=1}^{\infty} \frac{n!\cos(n!x)}{n^2}$$

div for any $x \in \mathbb{R}$

Is it a contradiction?

Ans: No. This is not a power series

Eg: $\sum_{n=0}^{\infty} n^2 (\frac{1}{2})^n = ?$

Sol $f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

$f'(x) = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n$

$f''(x) = \sum_{n=2}^{\infty} n(n-1) x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) x^n$

$n^2 = (n+1)(n+2) - 3(n+1) + 1$

$\therefore \sum_{n=0}^{\infty} n^2 x^n \Big|_{x=\frac{1}{2}} = f''(x) - 3f'(x) + f(x) \Big|_{x=\frac{1}{2}} = 16 - 12 + 2$

$= 6$

Method 2

$$xf'(x) = \sum_{n=1}^{\infty} nx^n = \sum_{n=0}^{\infty} nx^n$$

$$x^2 f''(x) = \sum_{n=2}^{\infty} n(n-1)x^n = \sum_{n=0}^{\infty} n(n-1)x^n$$

$$n^2 = n(n-1) + n$$

$$\sum_{n=0}^{\infty} n^2 x^n \Big|_{x=\frac{1}{2}} = x^2 f''(x) + xf'(x)$$

$$= 4 + 2 = 6$$

Term by term integration

Thm If $f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$

conv abs on $|x-a| < R$

Then: (1): $\sum_{n=0}^{\infty} \frac{C_n}{n+1} (x-a)^{n+1}$ $\left(\overset{\text{def}}{=} g(x)\right)$

also conv. abs. on $|x-a| < R$

(2): $(1) = \int_a^x f(t)\,dt$ $\left(\overset{\text{def}}{=} h(x)\right)$

$\because g'(x) = h'(x) = f(x)$, $g(a) = 0 = h(a)$

$\therefore g(x) = h(x)$

Eg $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

(1) $R = ?$ (2) $f(x) = ?$

(i) $\because f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

Ratio or root $\Rightarrow R = 1$

(ii) on $|x| < 1$,

$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots$

$\frac{1}{1+x^2} \longrightarrow = \cancel{1}\,(1+x^2)$

$f(x) = \int_0^x \frac{1}{1+t^2}\,dt = \tan^{-1} x$

Eg: $\ln(1\pm x)$. $|x|<1$

$$\frac{1}{1\pm x} = 1 \mp x + x^2 \mp x^3 + \cdots$$

$$\int_0^x \frac{1}{1\pm t}\,dt = x \mp \frac{x^2}{2} + \frac{x^3}{3} \mp \frac{x^4}{4} + \cdots$$

$$\pm \ln|1\pm t|\Big|_0^x \underset{|x|<1}{=\!=} \pm\ln(1\pm t)\Big|_0^x = \pm\ln(1\pm x)$$

$$\Rightarrow \ln(1\pm x) = \begin{cases} x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\ -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \end{cases}$$

$|x|<1$