Def Power Series:

$$\sum_{n=0}^{\infty} C_n (x-a)^n$$

$a$: center, $C_n$: coefficients

$x=a \Longrightarrow$ convergent.

Key issue: for what values of $x \in \mathbb{R}$, does the power series converge?

Eg. 1: $\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n (x-2)^n$

This is a Geometric Series

$$r = \frac{-1}{2}(x-2).$$

Convergent $\iff |r| < 1$

$\iff 0 < x < 4$

Remark: It converges absolutely on $0 < x < 4$

and diverges $\begin{pmatrix}\text{not cond.}\\ \text{conv.}\end{pmatrix}$ elsewhere

Eg2 $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Ratio test on

$|u_n| = \frac{|x|^n}{n!}$

$\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$

conv. for all $x \in \mathbb{R}$

**Thm** If $\sum_{n=0}^{\infty} a_n x^n$ conv

$(x-a)^n$

at $x = c \neq 0$ $(\neq a)$ then it

conv. absolutely for $|x| < |c|$

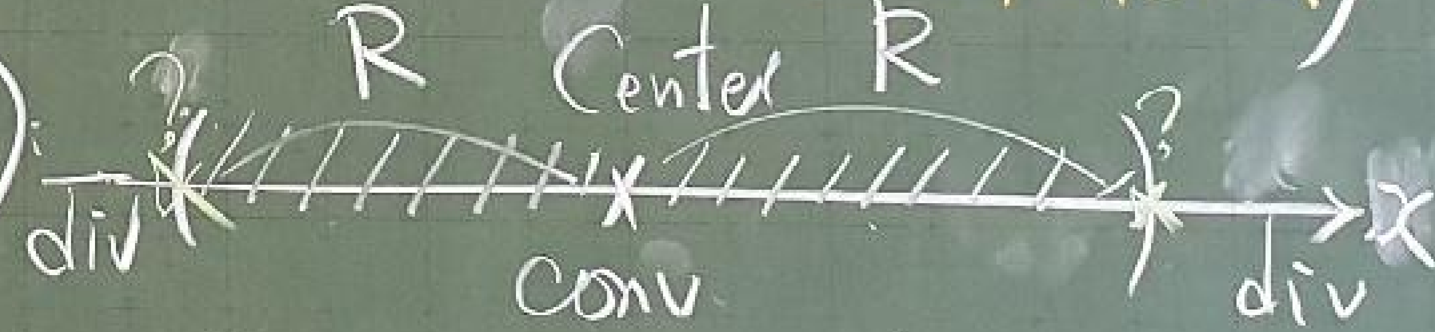
$|x-a| < |c-a|$

($\Rightarrow$ If it diverges at $x = d$

then it div. for $|x| > |d|$

$|x-a| > |d-a|$)

(O):

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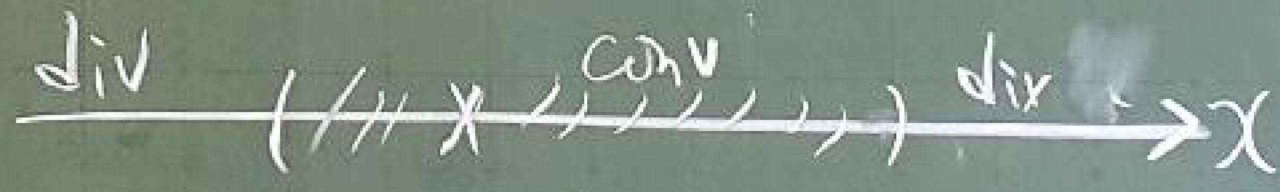


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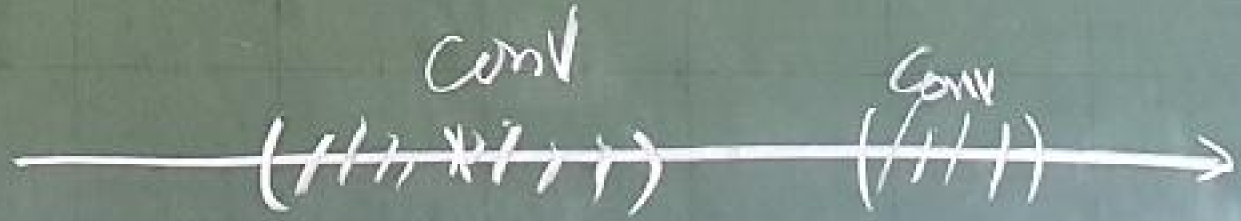

pf: If $\sum_{n=0}^{\infty} a_n c^n$ conv

$$\Rightarrow \lim_{n\to\infty} a_n c^n = 0$$

$$\Rightarrow |a_n c^n| < 1 \quad \forall n \geq N$$

$$\Rightarrow |a_n| < \frac{1}{|c|^n} \quad \forall n \geq n$$

If $|x| < |c|$

$$\sum_{n=N}^{\infty} |a_n x^n| < \sum_{n=N}^{\infty} \left(\frac{|x|}{|c|}\right)^n < \infty$$

i.e. conv abs. $|x| < |c|$

(absolutely)

Possible cases for $\sum_{n=0}^{\infty} c_n(x-a)^n$

(1) Conv abs. for all $x \in \mathbb{R}$

("Radius of conv" $= \infty$)

(2) conv only for $x = a$

("Radius of conv" $= 0$)

(3) $\exists\ 0 < R < \infty$, such that it $\begin{cases} \text{conv (absolutely)} & |x-a| < R \\ \text{div} & |x-a| > R \end{cases}$

("Radius of conv." = R)

(no conclusion on $x - a = \pm R$)

i.e. The radius of conv. for $\sum_{n=0}^{\infty} c_n(x-a)^n$ is R if the series conv. (abs.) on $|x-a|<R$ and div. on $|x-a|>R$

Remark: If $0 < R < \infty$

Then $\sum_{n=0}^{\infty} a_n (x-a)^n$ can either conv. or div at $x-a = \pm R$

Eg: (a) $\sum_{n=1}^{\infty} x^n$ conv on $(-1, 1)$
div elsewhere

(b) $\sum_{n=1}^{\infty} \frac{x^n}{n}$ conv on $[-1, 1)$
div elsewhere

(c) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ conv on $[-1, 1]$
div elsewhere

Ratio test on $|u_n| = |a_n (x-a)^n|$

If $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \tilde{\rho}$ $(0 \leq \tilde{\rho} \leq \infty)$ exists

Then $R = \dfrac{1}{\tilde{\rho}}$

$$\because \lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = |x-a| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

$$\lessgtr 1 \text{ if } |x-a| \lessgtr \frac{1}{\tilde{\rho}} \begin{pmatrix} \text{conv} \\ \text{div} \end{pmatrix}$$

Similarly, if $\displaystyle\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \tilde{\rho}$ exists in $[0,\infty]$, then $R = \dfrac{1}{\tilde{\rho}}$

<u>Remark</u>: Radius of conv.

$R \in [0, \infty]$ always exists.

but $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ or $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

need not exist.

<u>Eg</u>: $\sum_{n=1}^{\infty} a_n x^n$

$$= \left(\frac{x}{2}\right)^1 + \left(\frac{x}{4}\right)^2 + \left(\frac{x}{2}\right)^3 + \left(\frac{x}{4}\right)^4 + \cdots$$

Remark: (Beyond this course)

$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$ always exists in $[0,\infty]$!

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}}$$

In previous example

$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{2}$, $R=2$

(Note: $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \infty$)

Algebraic operations

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$

$B(x) = \sum_{n=0}^{\infty} b_n x^n$

both conv. abs. on $|x|<R$

$A(x) \cdot B(x) = ?$

$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots$

$\Rightarrow A(x) B(x) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k b_{n-k} \right) x^n$

$= a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2$

$+ \cdots$

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Thm: If $A(x)=\sum_{n=0}^{\infty} a_n x^n$, $B(x)=\sum_{n=0}^{\infty} b_n x^n$

both conv absolutely on $|x|<R$

and $C_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$

Then $\sum_{n=0}^{\infty} C_n x^n$ also conv. abs.

on $|x|<R$ and $= A(x)\cdot B(x)$

<u>Reason</u>: If $\lim |a_n|^{\frac{1}{n}} = \frac{1}{R_1}$, $R_1 \geq R$

$\lim |b_n|^{\frac{1}{n}} = \frac{1}{R_2}$, $R_2 \geq R$

$\Rightarrow |a_k| \sim \left(\frac{1}{R_1}\right)^k$, $|b_{n-k}| \sim \left(\frac{1}{R_2}\right)^{n-k}$

$\Rightarrow |C_n| \leq (n+1)\left(\frac{1}{R}\right)^n \Rightarrow \lim |C_n|^{\frac{1}{n}} \leq \frac{1}{R}$

$R_1 \geq R$

$\frac{1}{R_1} \leq \frac{1}{R}$

$|a_k| \sim \left(\frac{1}{R_1}\right)^k \leq \left(\frac{1}{R}\right)^k$

$|b_{n-k}| \quad \leq \left(\frac{1}{R}\right)^{n-k}$

How about Power Series
for $A(x)/B(x)$, if it exists?

$\underline{\text{Sol.}}$ Assume $\dfrac{A(x)}{B(x)} = \sum_{n=0}^{\infty} c_n x^n$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = \left(\sum_{n=0}^{\infty} b_n x^n\right)\left(\sum_{n=0}^{\infty} c_n x^n\right)$$

$$a_0 = b_0 \underline{c_0} \Rightarrow c_0 = \frac{a_0}{b_0}$$

未知數

$$a_1 = b_0 \underline{c_1} + b_1 c_0 \Rightarrow c_1 = \frac{a_1 - b_1 c_0}{b_0}$$

$$a_2 = b_0 \underline{c_2} + b_1 c_1 + b_2 c_0 \Rightarrow c_2 = \frac{***}{b_0}$$

and then solve for a_3, a_4 and so on ...