Alternating Series

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n, \quad u_n > 0$$

Eg: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

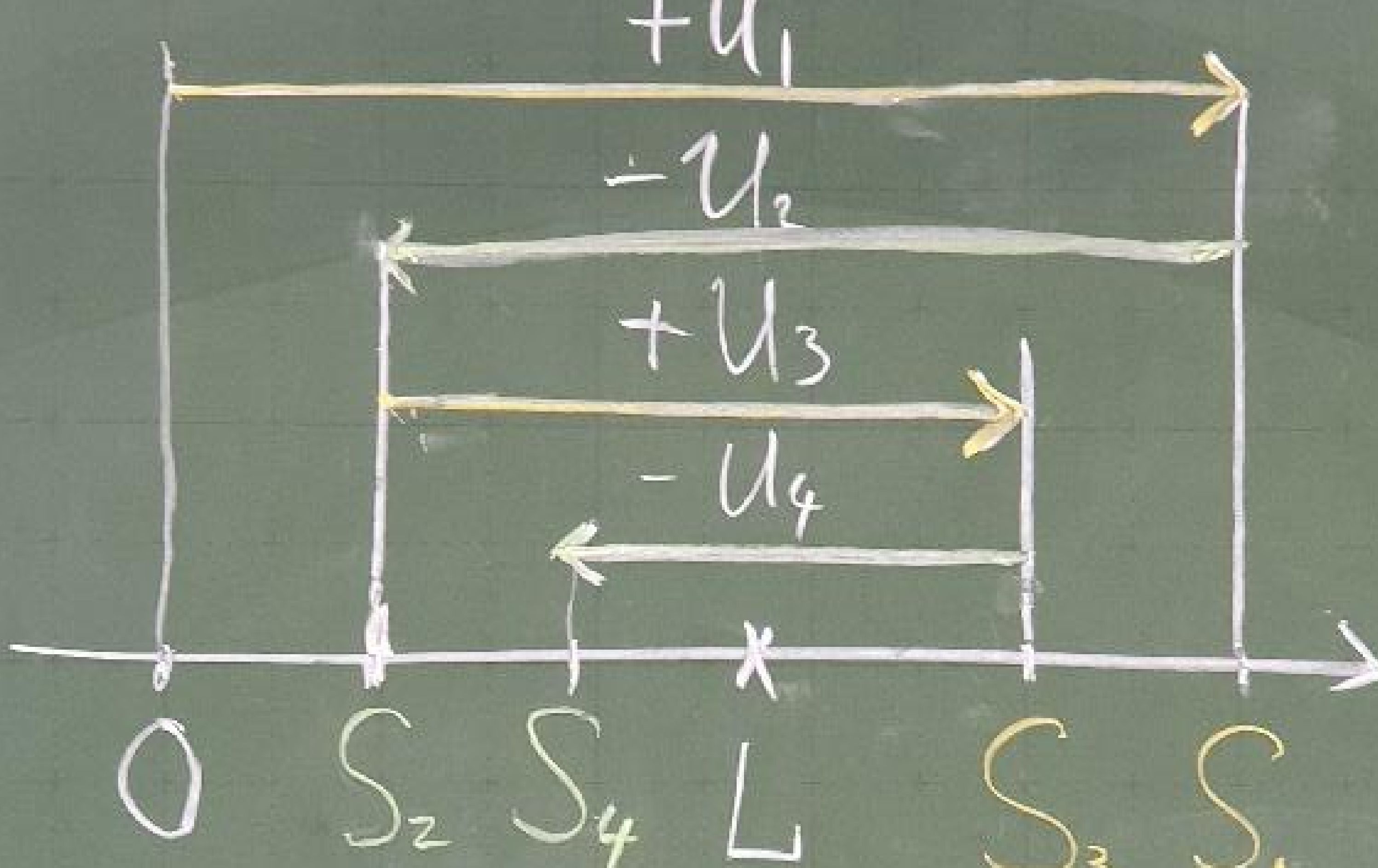
Leibnitz Test:

If (1) $u_n > 0$

(2) $u_n \geq u_{n+1}$ for all $n \geq N$.

(3) $\lim_{n \to \infty} u_n = 0$

Then $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges pf: Let $S_n = \sum_{k=1}^{n} (-1)^{k+1} u_k$



(1) $S_2 < S_4 < \cdots < S_1$

(2) $S_1 > S_3 > \cdots > 0$

$\therefore$ (1) is an increasing sequence bounded above

$$\therefore \lim_{k\to\infty} S_{2k} = L$$

$$S_{2k+1} = S_{2k} + u_{2k+1}$$

$$\xrightarrow{k\to\infty} L + 0$$

$$\therefore \lim_{n\to\infty} S_n = L$$

Remarks

(1) $S_{2m} < L < S_{2n+1}$

for any $m, n \in \mathbb{N}$

(2) $0 < L - S_{2k} < u_{2k+1}$

$u_{2k} > S_{2k-1} - L > 0$

$\Rightarrow |L - S_n| < u_{n+1}$

Def $\sum_{n=1}^{\infty} a_n$ converges absolutely

if $\sum_{n=1}^{\infty} |a_n| < \infty$

Def $\sum_{n=1}^{\infty} a_n$ Converges Conditionally

if $\begin{cases} \sum_{n=1}^{\infty} a_n \text{ converges} \\ \sum_{n=1}^{\infty} |a_n| = \infty \end{cases}$

Eg (a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots$

converges conditionally

(b) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges abs.

Thm: If $\sum_{n=1}^{\infty} |a_n|$ converges

Then $\sum_{n=1}^{\infty} a_n$ converges.

pf: $-|a_n| \le a_n \le |a_n|$

$$0 \le a_n + |a_n| \le 2|a_n|$$

$$S_N = \sum_{n=1}^{N} (a_n + |a_n|) \le 2 \sum_{n=1}^{N} |a_n| \le 2 \underline{\sum_{n=1}^{\infty} |a_n|}$$

$$a_N + |a_N| \ge 0$$

$$S_1 \le S_2 \cdots \le S_N \le \cdots \le M$$

$$\therefore \sum_{n=1}^{\infty} a_n + |a_n| = \lim_{N \to \infty} S_N \text{ converges}$$

$$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(a_n+|a_n|) - \sum_{n=1}^{\infty}|a_n|$$

$$= (\text{conv. Series}) - (\text{conv. series})$$

$$= \text{Convergent} \qquad \square$$

Eg (b): $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges abs.

$\Rightarrow$ converges.

Eg (c): $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$, $p>0$.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} <\infty & p>1 \\ =\infty & 0<p\le 1 \end{cases}$$

$$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} \text{ conv.} \begin{cases} \text{abs.} & p>1 \\ \text{cond.} & 0<p\le 1 \end{cases}$$

Eg (d): $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{\ln n}}$ conv? abs? cond?

$u_n = \dfrac{1}{n\sqrt{\ln n}}$ satisfies the requirements of Leibnitz test

$\therefore \displaystyle\sum_{n=2}^{\infty} (-1)^{n+1} u_n$ converges.

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}} \longrightarrow \text{check} \int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx$$

$$\lim_{b\to\infty} \int_2^b \frac{1}{x\sqrt{\ln x}}\,dx \;\underset{dx = e^y dy}{\overset{x = e^y}{=\!=}}\; \lim_{b\to\infty} \int_{y=\ln 2}^{\ln b} y^{-\frac{1}{2}}\,dy$$

$\Rightarrow$ diverges. $\therefore$ <u>conv. conditionally</u>

Remark:

increases $\longleftarrow \sum_{n=2}^{\infty} n^{-p}$

$\sum_{n=2}^{\infty} n^{-p}$ div $\qquad \sum_{n=2}^{\infty} n^{-p}$ conv.

$0<p\le 1 \qquad p=1 \qquad p>1 \qquad p$

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$$

No suitable Series to compare with.

Alternatively,

$\sum_{n=2}^{\infty} n^{q}$ increases

q increases

$\sum_{n=2}^{\infty} n^{q}$ 收斂

$\sum_{n=2}^{\infty} n^{q}$ 發散

$q<-1$

$q=-1$

$q\geq -1$

q

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$$

$$\sum_{n=2}^{\infty} \frac{1}{n}$$

Remark Root and Ratio test for $\sum_{n=1}^{\infty} a_n$, $a_n \neq 0$

Ratio test: (Similarly for root test)

Let $\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

(1) $0 \le \rho < 1$

$\Rightarrow \sum_{n=1}^{\infty} |a_n| < \infty$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges

[$\rho = 1$ no conclusion]

(2) $\rho > 1 \Rightarrow \lim_{n\to\infty} |a_n| \neq 0$

$\Rightarrow \lim_{n\to\infty} a_n \neq 0$ $\therefore$ divergent