Thm 10 Comparison Test.

Thm 11 Limit Comparison Test

$a_n > 0,\ b_n > 0$

① If $\lim_{n\to\infty} \frac{a_n}{b_n} = C,\ 0 < C < \infty$

Then $\sum_{n=1}^{\infty} a_n < \infty \Leftrightarrow \sum_{n=1}^{\infty} b_n < \infty$

② If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$

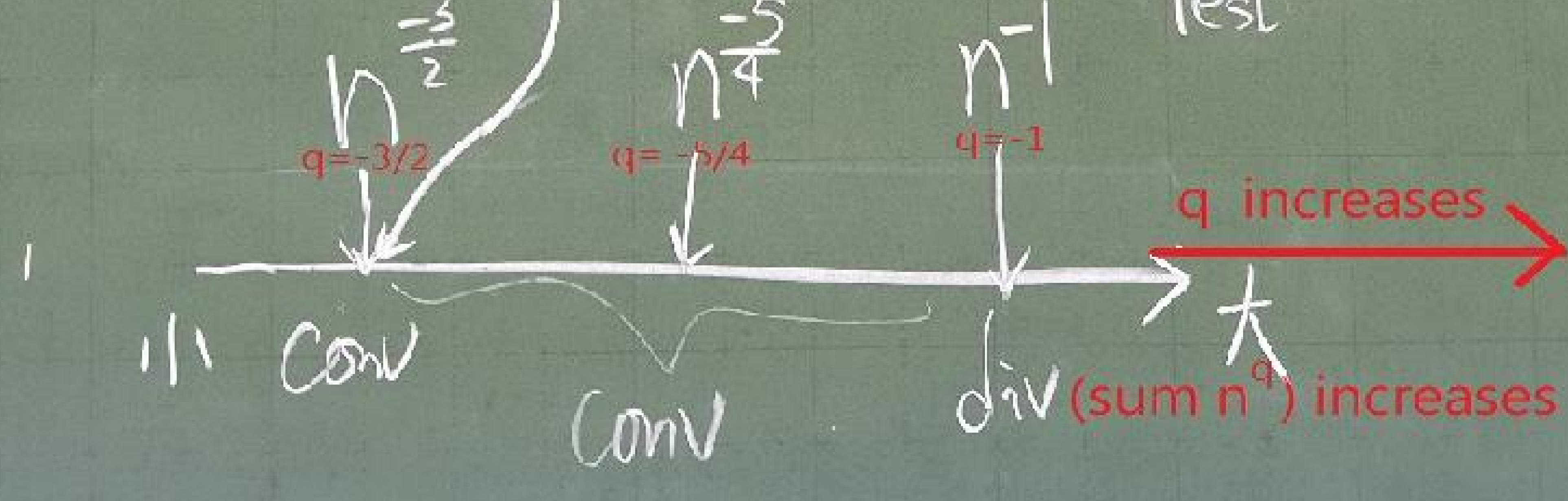
Then $\sum_{n=1}^{\infty} b_n < \infty \Rightarrow \sum_{n=1}^{\infty} a_n < \infty$

③ If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$

Then $\sum_{n=1}^{\infty} b_n = \infty \Rightarrow \sum_{n=1}^{\infty} a_n = \infty$

Eg 1

	Compare with	Thm	Result
ⓐ $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{(n+1)^2}$	$\displaystyle\sum \frac{1}{n}$	$\underline{10}$ $\underline{11}$	div.
ⓑ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$	$\displaystyle\sum \frac{1}{2^n}$	$\underline{11}$	conv.
ⓒ $\displaystyle\sum_{n=1}^{\infty} \frac{1+n\ln n}{n^2+5}$	$\displaystyle\sum \frac{\ln n}{n}$	$\underline{11}$	div.
ⓓ $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{\underbrace{n^{\frac{3}{2}}}}$	$\displaystyle\sum n^{\frac{-\frac{3}{2}+(-1)}{2}}$	$\underline{10}$ $\underline{11}$ Integral Test	conv.



Compare Thm Res

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ $\quad \displaystyle\sum \frac{1}{n}$ $\quad \underline{11}$ div.

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$ $\quad \displaystyle\sum n^{-\frac{3}{2}}$ $\quad \underline{11}$ conv

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{\frac{3}{2}}}$ $\quad \displaystyle\sum n^{-\frac{5}{4}}$ $\quad$ 10 11 conv.

$$\lim_{n\to\infty} \frac{(\ln n)^2 n^{-\frac{3}{2}}}{n^{-\frac{5}{4}}} = \lim_{n\to\infty} \frac{(\ln n)^2}{n^{\frac{1}{4}}} = \lim_{n\to\infty} \left(\frac{\ln n}{n^{\frac{1}{8}}}\right)^2 = 0$$

Compare Thm Res

(h) $\sum_{n=1}^{\infty} \tan\frac{1}{n^2}$ $\quad \sum \frac{1}{n^2}$ $\underline{\underline{11}}$ conv.

$$\lim_{n\to\infty} \frac{\tan\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{x\to 0^+} \frac{\tan x}{x} = 1$$

(i) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$ $\quad \sum \frac{1}{n}$ $\underline{\underline{11}}$ div.

$$\lim_{n\to\infty} \frac{n^{-1}}{\frac{1}{n\sqrt[n]{n}}} = \lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} e^{\left(\frac{\ln n}{n}\right)}$$

$$= e^{\lim\left(\frac{\ln n}{n}\right)} \overset{\text{L'Hôpital}}{=\!=} e^0 = 1, \sum \frac{1}{n} = \infty$$

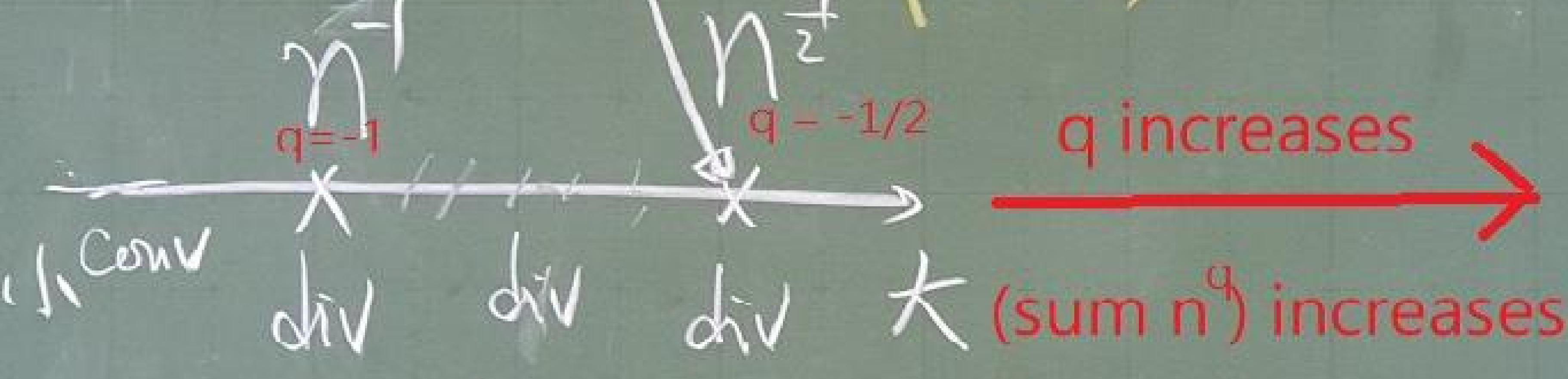
$\therefore$ By Limit Comp. Thm $\Rightarrow \sum \frac{1}{n\sqrt[n]{n}} = \infty$

Compare Thm Res

① $\sum_{n=1}^{\infty}\sqrt{\frac{\ln n}{n}} > \sum \frac{1}{n^{\frac{1}{2}}}$ $\underline{\underline{10}}$ div

② $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n \ln n}}$ $\sum n^{\frac{-3}{4}}$ $\underline{\underline{\frac{10}{11}}}$ div

$(\sum n^{-1})$



$$\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n \ln n}}}{n^{\frac{-3}{4}}} = \lim_{n\to\infty}\frac{n^{\frac{1}{4}}}{(\ln n)^{\frac{1}{2}}} = \infty$$

$$\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n \ln n}}}{n^{-1}} = \lim_{n\to\infty}\frac{n^{\frac{1}{2}}}{(\ln n)^{\frac{1}{2}}} = \infty$$

Ratio Test:

If $a_n > 0$ $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$

(a) $0 \leq \rho < 1 \Rightarrow \sum a_n < \infty$

(b) $\rho > 1 \Rightarrow \sum a_n = \infty$

(c) $\rho = 1$: Inconclusive.

Eg: $\begin{cases} \sum \frac{1}{n} = \infty, & (\rho = 1) \\ \sum \frac{1}{n^2} < \infty, & (\rho = 1) \end{cases}$

pf: ⓐ If $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$, $0<\rho<1$

Then Compare it with $\sum r^n$

$\because$ $0<\rho<1$. Let $\underbrace{r = \frac{1+\rho}{2}}$, $0<r<1$.

$\therefore$ $\sum r^n < \infty$

$\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$. Take $\varepsilon = r-\rho > 0$

Then $\exists\, N>0$ such that

$$n \geq N \Rightarrow \left|\frac{a_{n+1}}{a_n} - \rho\right| < \varepsilon$$

Therefore $n > N \Rightarrow \frac{a_{n+1}}{a_n} < \rho + \varepsilon = r$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

$$\leq \sum_{n=1}^{N} a_n + a_{N+1} + a_{N+2} + \cdots$$

$$\leq \sum_{n=1}^{N} a_n + a_N r + a_N r^2 + \cdots$$

$$= \sum_{n=1}^{N} a_n + \left(\begin{array}{c}\text{convergent} \\ \text{series}\end{array}\right) < \infty$$

(b) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho > 1$

Then $\exists\, N > 0$ such that

$$n \geq N \Rightarrow \frac{a_{n+1}}{a_n} > \frac{\rho+1}{2} > 1$$

$\lim_{n\to\infty} a_n \neq 0 \quad \therefore \sum_{n=1} a_n = \infty$

Root test

If $a_n > 0$, $\lim_{n\to\infty} a_n^{\frac{1}{n}} = \rho$

(a) $0 \le \rho < 1 \Rightarrow \sum a_n < \infty$

(b) $\rho > 1 \Rightarrow \sum a_n = \infty$

(c) $\rho = 1$: Inconclusive

Eg: $\begin{cases} \sum \frac{1}{n} = \infty \ (\rho = 1) \\ \sum \frac{1}{n^2} < \infty \ (\rho = 1) \end{cases}$

Eg (i) $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$

root test: $\lim_{n\to\infty} \left(\frac{2^n+5}{3^n}\right)^{\frac{1}{n}} = \frac{2}{3}$.

(ii) $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$

Ratio test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} =$

$= \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)(n+1)} = 4$, div

(iii) $\sum_{n=1}^{\infty} \frac{4^n n! n!}{(2n)!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

$$= \lim_{n\to\infty} \frac{4(n+1)(n+1)}{(2n+1)(2n+2)} = 1$$

$\Rightarrow$ No conclusion yet

However.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)(n+1)}{(n+\frac{1}{2})(n+1)} > 1$$

$$0 < \cdots < a_n < a_{n+1} < \cdots$$

$\therefore \lim_{n\to\infty} a_n \neq 0$, div

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Method 1: Compare with $\displaystyle\sum \frac{1}{(\frac{3}{2})^n}$

Method 2: Root test

$$\lim_{n\to\infty} \left(\frac{n^2}{2^n}\right)^{\frac{1}{n}} = \frac{1}{2}, \text{ conv}$$

Method 3 Ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{n+1}{n}\right)^2}{2} = \frac{1}{2}$$

conv

$\sum a_n$

(v) $a_n = \begin{cases} \dfrac{n}{2^n}, & n = \text{odd} \\ \dfrac{1}{2^n}, & n = \text{even} \end{cases}$

$\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$ does not exist

But $\lim\limits_{n\to\infty} a_n^{\frac{1}{n}} = \dfrac{1}{2}$, conv.

(vi) $\sum\limits_{n=1}^{\infty} \dfrac{2^n}{n^3}$, root: $\rho = 2$, div

(vii) $\sum\limits_{n=1}^{\infty} \left(\dfrac{1}{1+n}\right)^n$, root: $\rho = 0$, conv