Why do we care about infinite sequence and series?

Eg: $\int_0^1 (1+x^3)^{\frac{1}{2}}\, dx = ?$

Eg: Solve $y'' - 2ty' - 2y = 0$

Possible solution method:

Given $f(x)$, can we find $a_n \in \mathbb{R}$, such that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \; ?$$

It can be shown that if a_n exists, then

$$a_n = \frac{f^{(n)}(0)}{n!}$$

Questions: is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^{n}$$

for $x \neq 0$?

Is it true that

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} ?$$

$$\int_0^x f(t)\,dt \overset{?}{=} \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

Fundamental Questions

Sequence $a_0, a_1, \dots a_n \dots$

Series $\sum_{n=1}^{\infty} a_n$

How to define

$$\lim_{n \to \infty} a_n = L$$

and

$$\sum_{n=1}^{\infty} a_n = L \;?$$

Infinite Series

Def: $\sum_{n=1}^{\infty} a_n = L$

iff $\lim_{n\to\infty} S_n = L$

where $S_n = \sum_{k=1}^{n} a_k$

Remark: If $\sum_{n=1}^{\infty} a_n = \begin{matrix} +\infty \\ -\infty \end{matrix}$

we also call it divergent.

Example of convergent Series

Geometric Series.

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

$$S_n = a + ar + \cdots + ar^{n-1}$$

$$rS_n = ar + \cdots + ar^{n-1} + ar^n$$

$$(1-r)S_n = a(1-r^n)$$

$$r \neq 1 \Rightarrow S_n = \frac{a(1-r^n)}{1-r} \begin{cases} \text{conv: } |r| < 1 \\ \text{div } |r| > 1 \\ \quad r = -1 \end{cases}$$

$r = 1$: $a + a + a + \cdots$ divergent.

Thm 7.

$$\sum_{n=1}^{\infty} a_n \text{ conv.} \underset{(\not\Leftarrow)}{\Longrightarrow} \lim_{n\to\infty} a_n = 0$$

$$\left(\sum_{n=1}^{\infty} a_n \text{ div.} \Longleftarrow \lim_{n\to\infty} a_n \neq 0 \right)$$

Pf: Since $a_n = S_n - S_{n-1}$

If $\lim_{n\to\infty} S_n = S$, then $\lim_{n\to\infty} S_{n-1} = S$

$$\Rightarrow \lim_{n\to\infty} a_n = S - S = 0$$

Examples of divergent Series

Eg 1 ($\lim a_n \neq 0$)

$$\sum_{n=1}^{\infty} n^2, \quad \sum_{n=1}^{\infty} (-1)^n, \quad \sum_{n=1}^{\infty} \frac{-n}{2n+5}$$

Eg 2: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ ($\divideontimes$)

Sol: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots$

$> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\cdots$

$= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots\cdots$ divergent

Eg 3: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Sol: $S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$

$= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right)$

$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right)$
$+ \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$= 1 - \frac{1}{n+1}$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n \to \infty} S_n = 1.$

How about $\sum_{n=1}^{\infty} \frac{1}{n^p}$? $(p>0)$ pf

Integral Test:

If $a_n > 0$. $a_n = f(n)$

and $f(x)$ is positive, cont.

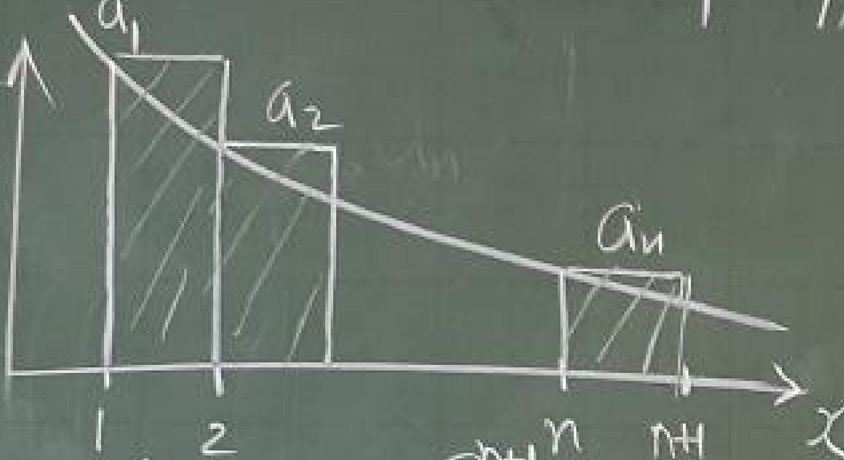
and decreasing.

for all $x \geq N$ $\left(\begin{matrix} N: \text{positive} \\ \text{integer} \end{matrix}\right)$

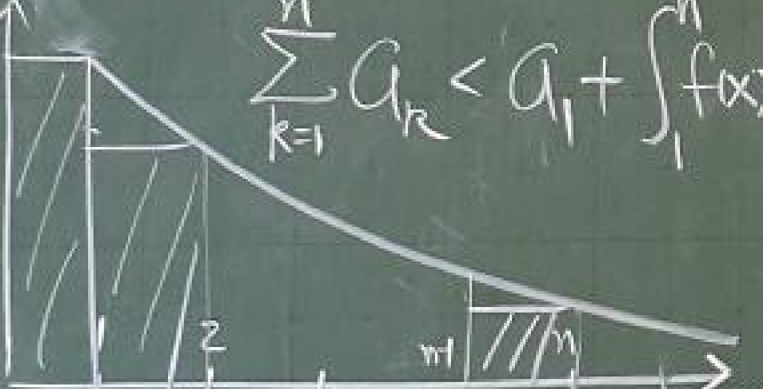
Then $\sum_{n=N}^{\infty} a_n < \infty \Leftrightarrow \int_N^{\infty} f(x)\,dx < \infty$

$\left(\text{Eg}: \sum_{n=1}^{\infty} \frac{1}{n^p} : \text{check } \int_1^{\infty} \frac{1}{x^p}\,dx \right)$

pf: (Take N=1 for simplicity)



$$\Rightarrow \sum_{k=1}^{n} a_k > \int_1^{n+1} f(x)\,dx$$



$$\sum_{k=1}^{n} a_k < a_1 + \int_1^{n} f(x)\,dx$$

$$\underset{(*_1)}{\int_1^{n+1} < \sum_{k=1}^{n} a_k} < \underset{(*_2)}{a_1 + \int_1^{n}}$$

Thm 6 (from advanced calculus)

If $S_1 \le S_2 \le \cdots \le S_n \le \cdots \le M$.

Then $\lim_{n\to\infty} S_n$ exists (and $\le M$).

(i) $S_{n-1} < S_n = \sum_{k=1}^{n} \overset{(*_2)}{<} a_1 + \int_1^n < \underbrace{a_1 + \int_1^\infty f(x)dx}_{M}$

$\therefore \int_1^\infty f(x)dx < \infty \xrightarrow[(M = a_1 + \int_1^\infty)]{\text{Thm 6}} \lim_{n\to\infty} S_n < \infty$

(ii) If $\int_1^\infty f(x)dx = \infty$

$\Rightarrow \lim_{n\to\infty} \int_1^{n+1} = \infty \overset{(*_1)}{\Rightarrow} \lim_{n\to\infty} \left(\sum_{k=1}^{n} a_k \right) = \infty$

Eg 4. $\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} = \infty & 0 < p \leq 1 \\ < \infty & p > 1 \end{cases}$

Eg 5: $\sum_{n=0}^{\infty} \frac{1}{1+n^2} < \infty$

$\lim_{b \to \infty} \int_0^b \frac{1}{1+x^2} dx = \lim_{b \to \infty} \tan^{-1} b - \tan^{-1} 0 = \frac{\pi}{2}$

Eg 6 $\sum_{n=1}^{\infty} n e^{-n^2} < \infty$

$\because \lim_{b \to \infty} \int_1^b x e^{-x^2} dx = \lim_{b \to \infty} \frac{1}{2} \left(e^{-b^2} - e^{-1} \right) < \infty$

Eg 7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{2^{\ln n}} = \infty$

$\ln x = y$

Sol: $\displaystyle\lim_{b\to\infty} \int_2^b \frac{1}{2^{\ln x}}\,dx \quad \begin{pmatrix} x = e^y \\ dx = e^y\,dy \end{pmatrix}$

$\displaystyle = \lim_{b\to\infty} \int_{y=\ln 2}^{\ln b} \frac{1}{2^y} e^y\,dy$

$\displaystyle = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \left(\frac{e}{2}\right)^y dy = \infty$

$\left(\frac{e}{2} > 1\right)$

Comparison Test: $\sum_{n=1}^{\infty} a_n$ conv?

If $0 < d_n \leq a_n \leq c_n$

for all $n \geq N$

Then

① $\sum_{n=1}^{\infty} c_n < \infty \Rightarrow \sum_{n=1}^{\infty} a_n < \infty$

② $\sum_{n=1}^{\infty} d_n = \infty \Rightarrow \sum_{n=1}^{\infty} a_n = \infty$

Eg 8 $\sum_{n=1}^{\infty} \frac{5}{5n-1}$

$$\sum_{n=1}^{\infty} \frac{5}{5n-1} \geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

<u>Eg 9</u> $\sum_{n=0}^{\infty} \frac{1}{n!} < \infty$

<u>Sol</u> $= 1 + \frac{1}{1} + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2 \cdot 1} + \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} + \cdots$

$$< 1 + \frac{1}{1} + \frac{1}{2 \cdot 1} + \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 2} + \cdots$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots < \infty$$

$$\therefore \sum_{n=0}^{\infty} \frac{1}{n!} < \infty$$

Limit comparison Test

If $a_n > 0,\ b_n > 0\ \forall n \geq N$

(i) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c \quad 0 < c < \infty$

Then $\sum\limits_{n=1}^{\infty} b_n < \infty \underset{(=\infty)}{} \Leftrightarrow \sum\limits_{n=1}^{\infty} a_n < \infty \underset{(=\infty)}{}$

(ii) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$

Then $\sum\limits_{n=1}^{\infty} b_n < \infty \Rightarrow \sum\limits_{n=1}^{\infty} a_n < \infty$

(iii) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$

Then $\sum\limits_{n=1}^{\infty} b_n = \infty \Rightarrow \sum\limits_{n=1}^{\infty} a_n = \infty$