<u>Thm 2</u> If f, g are cont. on $[a, \infty)$ and $0 \le f \le g$

Then

(1) $\int_a^\infty g(x)\,dx < \infty \Rightarrow \int_a^\infty f(x)\,dx < \infty$

(2) $\int_a^\infty f(x)\,dx = \infty \Rightarrow \int_a^\infty g(x)\,dx = \infty$

Similarly for $\lim_{x \to -\infty}$ and type II

Thm 3 If f, g are cont. on $[a,\infty)$ and $f>0$, $g>0$

and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$

(i) $0<L<\infty$

$$\int_0^\infty f(x)\,dx < \infty \Longleftrightarrow \int_0^\infty g(x)\,dx < \infty$$

(ii) $L=0$ " $\Longleftarrow$ "

(iii) $L=\infty$ " $\Longrightarrow$ "

Similarly for $\lim_{x\to-\infty}$ and type II

Eg 7: Check if the following converge.

	$\int_a^{\infty} f(x)\,dx$	$g(x)$	Thm
ⓐ	$\int_1^{\infty} \frac{\sin^2 x}{x^2}\,dx$	$\frac{1}{x^2}$	2-(1)
ⓑ	$\int_1^{\infty} \frac{1}{\sqrt{x^2+0.1}}\,dx$	$\frac{1}{x}$	3-(i)
ⓒ	$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{x}}\,dx$	$\frac{1}{\sqrt{x}}$	$\begin{cases} 2\text{-(1)} \\ 3\text{-(i)} \end{cases}$ $(x \to 0^+)$
ⓓ	$\int_1^{\infty} \frac{1-e^{-x}}{x}\,dx$	$\frac{1}{x}$	3-(i)

Eg 8: $\int_1^\infty \frac{1}{x(x^2-1)^{1/3}}\,dx$ converge?

Sol: Check $\int_1^2$ and $\int_2^\infty$

$$f(x) = \frac{1}{x(x^2-1)^{1/3}}$$

ⓐ $\int_1^2$: let $g(x) = \frac{1}{(x-1)^{1/3}}$ Thm 3-(i)

$$\lim_{x\to 1^+} \frac{f(x)}{g(x)} = 2^{-\frac{1}{3}}, \quad \int_1^2 g(x)\,dx < \infty \quad (p = \tfrac{1}{3})$$

$\therefore \int_1^2 f(x)\,dx < \infty$

(b). $\int_2^{\infty}$

Let $g(x) = \dfrac{1}{x^{1+\frac{2}{3}}}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{(x^2)^{\frac{1}{3}}}{(x^2-1)^{\frac{1}{3}}} = 1$$

$\therefore$ Thm 3-(i) $\Rightarrow \int_2^{\infty} f(x)\,dx < \infty$

(a), (b) $\Longrightarrow \int_1^{\infty} f(x)\,dx < \infty$

Eg 9: For what values of q does $\int_e^\infty x^q \ln x \, dx$ converge?

<u>Method 1:</u>

If $q \geq -1$, $\int_e^\infty x^q \ln x \, dx > \int_e^\infty x^q \, dx = \infty$

If $q < -1$

$$\lim_{x \to \infty} \frac{x^q \ln x}{x^{\frac{q+(-1)}{2}}} = 0 \quad \text{(l'Hôpital)}$$

$$\left[q < \frac{q+(-1)}{2} < -1 \right] \quad \int_e^\infty x^{\frac{q+(-1)}{2}} \, dx < \infty$$

From Thm 3 (ii)

$$\Rightarrow \int_e^\infty x^q \ln x \, dx < \infty$$

<u>Ans</u>: $q \geq -1$: $= \infty$

$q < -1$. $< \infty$.

Method 2:

Let $x = e^y$, $dx = e^y dy$

Sequence: $a_0, a_1, a_2, \cdots\cdots$

Def: $\lim_{n\to\infty} a_n = L$

Given any $\varepsilon > 0$, there exists a corresponding integer $N > 0$, such that

$$n > N \Longrightarrow |a_n - L| < \varepsilon$$

Def $\lim_{n\to\infty} a_n = \infty$ $-\infty$

Given any $M \in \mathbb{R}$ $m \in \mathbb{R}$, there exists a corresponding integer $N > 0$, such that

$$n > N \Rightarrow \begin{matrix} a_n > M \\ a_n < m \end{matrix}$$

Eg 1: $a_n = \{1, -1, 1, -1, \dots\}$

diverges (and $\neq \pm\infty$)

Eg 2: $a_n = \{1, 0, 2, 0, 3, 0, \dots\}$

diverges

Note $\lim\limits_{n\to\infty} a_n \neq +\infty$

$\neq -\infty$

$\neq$ any $L \in \mathbb{R}$

does not exist

Eg 3: $a_n = \{1, 4, 9, 16, \dots\}$

$$\lim_{n\to\infty} a_n = \infty$$

Eg 4 (Sandwich Thm)

$$\lim_{n\to\infty} \frac{\cos n}{n} = ?$$

Ans: $\frac{-1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$

$$\lim_{n\to\infty} \frac{-1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n}$$

$\therefore$ Ans $= 0$

Eg 5 $\lim_{n\to\infty} \frac{\ln n}{n} = ?$

Ans: find $\lim_{\substack{x\to\infty \\ (x\in\mathbb{R})}} \frac{\ln x}{x}$ $(=0)$

(l'Hôpital)

$\Rightarrow \lim_{n\to\infty} \frac{\ln n}{n} = 0$

Eg 6. $\lim\limits_{n\to\infty}\left(\frac{n+1}{n-1}\right)^n$

Sol (May think of $n\in\mathbb{R}$ as in previous examples)

$$= \lim_{n\to\infty} e^{n\ln\left(\frac{n+1}{n-1}\right)} \quad (= e^2)$$

$$\lim_{n\to\infty} n\ln\left(\frac{n+1}{n-1}\right) = \lim_{n\to\infty}\frac{\ln\left(\frac{1+\frac{1}{n}}{1-\frac{1}{n}}\right)}{\frac{1}{n}}$$

$$\overset{y=\frac{1}{n}}{=} \lim_{y\to 0^+}\frac{\ln(1+y)-\ln(1-y)}{y} = \lim_{y\to 0^+}\frac{\frac{1}{1+y}+\frac{1}{1-y}}{1} = 2$$