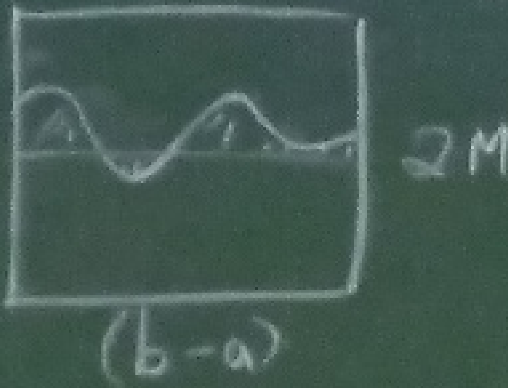
Improper integrals
(瑕積分)

Recall (proper) integrals

If $f(x)$ is cont. on $[a,b]$

$\Rightarrow |f(x)| \le M$ on $[a,b]$

$\Rightarrow \int_a^b f(x)\,dx$ exists (finite area)



If either $(b-a)$ or $\max|f|$ (or both) becomes ∞.

$\Longrightarrow$ Improper integrals

Def (II)

(1) f is cont. on $[a,\infty)$

$$\int_a^\infty f(x)dx \overset{\text{def}}{=} \lim_{b\to\infty}\int_a^b f(x)dx$$

(2) f is cont. on $(-\infty,b]$

$$\int_{-\infty}^b f(x)dx \overset{\text{def}}{=} \lim_{a\to-\infty}\int_a^b f(x)dx$$

(3) f is cont. on $(-\infty,\infty)$

$$\int_{-\infty}^\infty f(x)dx \overset{\text{def}}{=} \int_{-\infty}^c + \int_c^\infty \text{ for any } c\in\mathbb{R}$$

limits exist and are finite $\Longleftrightarrow$ converge

otherwise $\Longleftrightarrow$ diverge

In case (3)

$\int_{-\infty}^{\infty}$ converge if

both $\int_{-\infty}^{c}$ and $\int_{c}^{\infty}$ exist

and are finite.

If either $\int_{-\infty}^{c}$ or $\int_{c}^{\infty}$

diverges $\Longrightarrow \int_{-\infty}^{\infty}$ diverges

Remark If $\underline{f \geq 0}$

then $\int_a^\infty$, $\int_{-\infty}^b$ or $\int_{-\infty}^\infty$

either converge or diverge to ∞

We can write

$$\int_a^\infty f(x)\,dx = \infty$$

to mean it diverges

Eg 1 $\int_1^{\infty} \frac{\ln x}{x^2}\,dx = ?$

Sol $\int_1^{b} \frac{\ln x}{x^2}\,dx = \int_1^{b} \ln x \, d\left(\frac{-1}{x}\right)$

$= \frac{-1}{x}\ln x \Big|_1^b - \int_1^b \frac{-1}{x}\, d\ln x$

$= \frac{-1}{b}\ln b + 1 - \frac{1}{b}$

$\therefore \int_1^{\infty} = \lim_{b\to\infty} = 1$

Eg 2 $\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$ converge?

Sol: Can we do

$$\lim_{b\to\infty} \int_{-b}^{b} \frac{x}{1+x^2}\,dx \ ?$$

(No!)

$$= \lim_{b\to\infty} \frac{1}{2} \int_{-b}^{b} \frac{d(1+x^2)}{1+x^2}$$

$$= \lim_{b\to\infty} \frac{1}{2} \ln(1+x^2)\Big|_{-b}^{b} = 0$$

However $\int_0^\infty \frac{x}{1+x^2}\,dx$

$$= \lim_{b\to\infty}\int_0^b \frac{x}{1+x^2}\,dx = \lim_{b\to\infty}\frac{\ln(1+b^2)}{2} = \infty$$

conclusion: it diverges

Similarly. $\int_{-\infty}^0 \frac{x}{1+x^2}\,dx = -\infty$

Remark: $\lim_{b\to\infty}\int_{-b}^{2b}\frac{x}{1+x^2}\,dx$

$$= \lim_{b\to\infty}\frac{1}{2}\ln\frac{1+4b^2}{1+b^2} = \ln 2$$

This example explains why the integral $\int_{-\infty}^\infty$ has to be defined as in case (3) above.

Eg 3 $\int_1^\infty \frac{1}{x^p}\,dx \quad p>0$

Ans $p>1$: converge

$0<p\le 1$: diverge

$p\neq 1$, $\int_1^\infty x^{-p}\,dx = \lim_{b\to\infty} \frac{1}{-p+1} x^{-p+1}\Big|_1^b$

$p=1$, $\int_1^\infty x^{-1}\,dx = \lim_{b\to\infty} \ln x\Big|_1^b$

Def (II)

(1) If f is cont. on $(a,b]$

$$\int_a^b f(x)dx \stackrel{\text{def}}{=\!=} \lim_{c \to a^+} \int_c^b f(x)dx$$

(2) If f is cont. on $[a,b)$

$$\int_a^b f(x)dx \stackrel{\text{def}}{=\!=} \lim_{c \to b^-} \int_a^c f(x)dx$$

(3) If f is cont. on (a,b)

$$\int_a^b f(x)dx \stackrel{\text{def}}{=\!=} \int_a^c f(x)dx + \int_c^b f(x)dx$$

for any $c \in (a,b)$

(and any $c \in (a,b)$ will do)

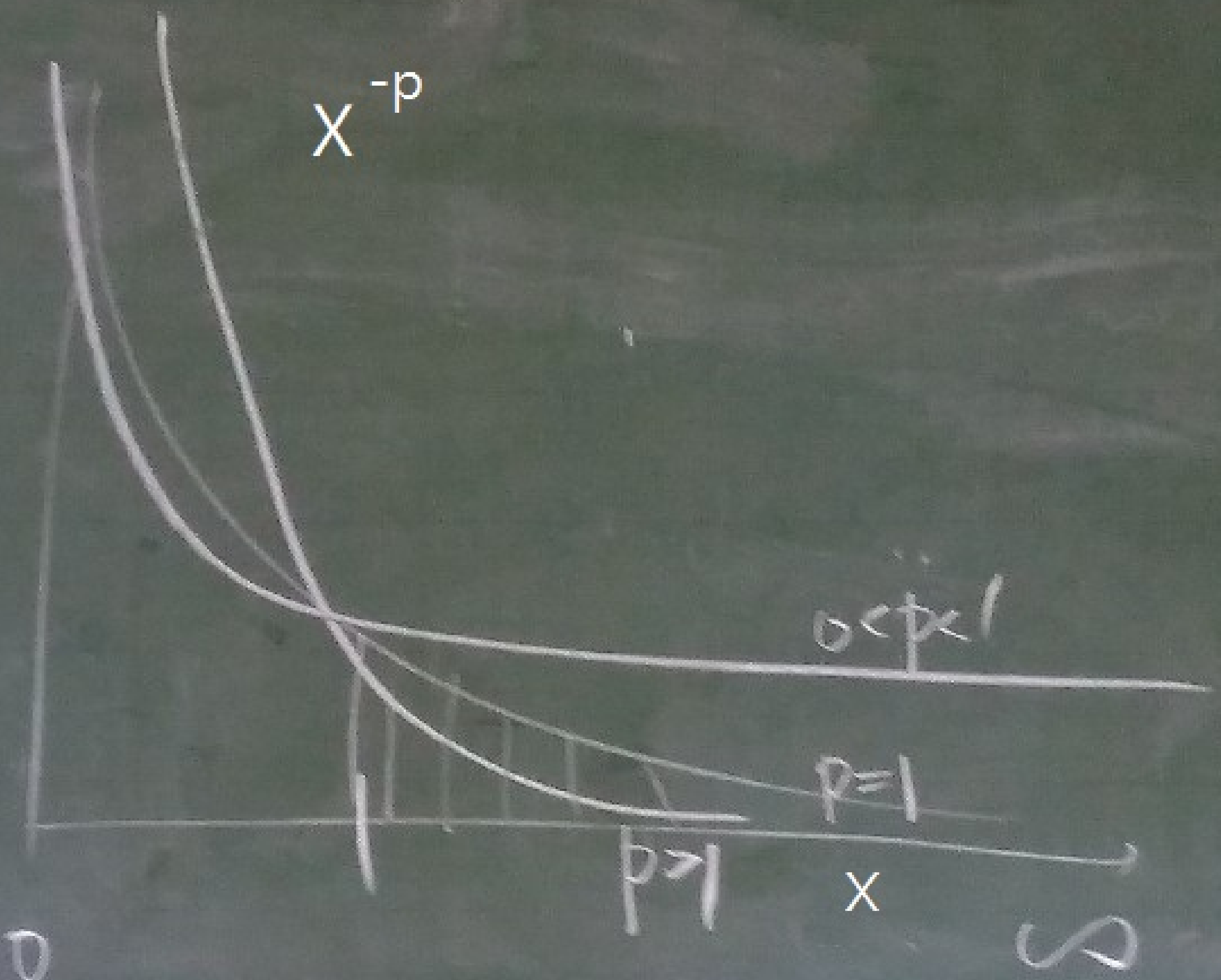
Eg 4 $\int_0^1 x^{-p}\,dx, \quad p>0$

Ans: $p \geq 1$: diverge

$0<p<1$: converge

$p \neq 1$. $\int_0^1 x^{-p}\,dx = \lim_{c\to 0^+} \frac{1}{-p+1} x^{-p+1}\Big|_c^1$

$p=1$. $\int_0^1 x^{-1}\,dx = \lim_{c\to 0^+} \ln x\Big|_c^1$



X -p
0<p<1
p=1
p>1
1
0
X
0
∞

Remark:

The same conclusion applies to $\displaystyle\int_c^{c+1} \frac{1}{(x-c)^p}\,dx$

and $\displaystyle\int_{c-1}^{c} \frac{1}{|x-c|^p}\,dx$

Hint: use change of variable $y = x - c$

Eg 5 $\int_0^1 \frac{1}{1-x}\,dx$ converge?

Ans: $p=1$. diverge
(use previous remark)

Sol: let $y=1-x$ $dy=-dx$

$$\int_{x=0}^{1} \frac{1}{1-x}\,dx = \int_{y=1}^{0} \frac{1}{y}(-dy)$$

$$= \int_{y=0}^{1} \frac{1}{y}\,dy \quad (p=1,\ \text{diverge})$$

Eg 6. $\int_0^3 \frac{dx}{(x-1)^{2/3}}$ converge?

Ans. $\int_0^3 = \underbrace{\int_0^1 + \int_1^3}$

both are of type II

From remark both $\int_0^1$ and $\int_1^3$ converge since $p = \frac{2}{3}$ (Exercise: $y = x-1$)

Eg 7. $\int_1^\infty e^{-x^2}dx$ converge?

$x>1.\ e^{-x^2} < e^{-x}$

$\therefore \int_1^\infty e^{-x^2}dx < \int_1^\infty e^{-x}dx = \frac{1}{e}$

$\therefore$ converge

Thm 2. If f, g are cont. on $[a, \infty)$ and $0 \le f \le g$

Then

(1) $\displaystyle\int_a^\infty g(x)\,dx < \infty \Rightarrow \int_a^\infty f(x)\,dx < \infty$

(conv) (conv)

(2) $\displaystyle\int_a^\infty f(x)\,dx = \infty \Rightarrow \int_a^\infty g(x)\,dx = \infty$

(div) (div)