Homework Assignment for Chapter 16

1. Section 16.1: Problems 15, 23, 25, 29.

2. Section 16.2: Problems 19, 25, 29(b), 35(a).

   See equation (5) in p932 and (6) in p934 for definition of flow, circulation and flux.

3. Section 16.3: Problems 1, 3, 5, 9, 21, 26, 29.

   Hint: For problems 1, 3, 5, use Component Test on p943. For problem 29: do not evaluate the line integrals directly. Try to find the potential function f for the vector field $\mathbf{F}$ instead.

4. Let $\boldsymbol{F} = \frac{x}{\sqrt{x^2+y^2}}\boldsymbol{i} + \frac{y}{\sqrt{x^2+y^2}}\boldsymbol{j} + 0\boldsymbol{k}$ and $\boldsymbol{G} = \frac{-y}{x^2+y^2}\boldsymbol{i} + \frac{x}{x^2+y^2}\boldsymbol{j} + 0\boldsymbol{k}$.

   (a) Show that both $\boldsymbol{F}$ and $\boldsymbol{G}$ satisfy the component test.

   (b) The natural domain for both $\boldsymbol{F}$ and $\boldsymbol{G}$ is $\{(x, y, z),\ x^2 + y^2 \neq 0\}$ (that is where $\boldsymbol{F}$ and $\boldsymbol{G}$ are defined). Show that $\boldsymbol{F}$ is conservative in this domain by finding its potential function.

   (c) Show that $\boldsymbol{G}$ is NOT conservative in this domain (see example 5 on p945).

   (d) If given another $\boldsymbol{H}$ satisfying the component test in this domain, how do you determine whether $\boldsymbol{H}$ is conservative?

5. Let $\boldsymbol{F} = \frac{x}{\sqrt{x^2+y^2+z^2}}\boldsymbol{i} + + \frac{y}{\sqrt{x^2+y^2+z^2}}\boldsymbol{j} + + \frac{z}{\sqrt{x^2+y^2+z^2}}\boldsymbol{k}$. What is the natural domain for $\boldsymbol{F}$? Show that $\boldsymbol{F}$ satisfies the component test in this domain. Is this domain simply connected? Is $\boldsymbol{F}$ conservative in this domain?

6. Section 16.4: Problems 10, 17, 19, 23, 27, 29, 38, 39.

7. Section 16.5: Problems 5, 11, 13, 19, 31, 33, 49, 51, 55, 56.

8. Section 16.6: Problems 17, 19, 21, 25, 35, 37.

9. Section 16.7: Problems 1, 3, 6, 7, 13, 23, 28.

10. This exercise is to show that Flux, Circulation and the Curl of a vector field does not depend on the coordinate you choose.

    Let x', y' be the coordinate axis obtained by rotating the x, y axis by a fixed angle θ.

    (a) Express x', y' in terms of x, y and vice versa.

    (b) Express $\frac{\partial}{\partial x'}$, $\frac{\partial}{\partial y'}$ in terms of $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and vice versa.

(c) Let (M, N) be the components of a vector filed $\boldsymbol{F}$ in the original (x, y) coordinate. Express the components of $\boldsymbol{F}$, (M', N') in the new (x', y') coordinates in terms of M and N.

(d) Use chain rule to verify that

$$\frac{\partial N'}{\partial x'} - \frac{\partial M'}{\partial y'} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

and

$$\frac{\partial M'}{\partial x'} + \frac{\partial N'}{\partial y'} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

(e) Express the unit vectors $\hat{x}'$, $\hat{y}'$ in terms of $\hat{x}$, $\hat{y}$ and vice versa.

(f) Let x', y' be defined as above. In 3D, we perform the change of variable from (x, y, z) to (x', y', z) (z coordinate is unchanged). Let $(M(x, y, z), N(x, y, z), (P(x, y, z))$ be the components of a vector field $\boldsymbol{F}$ in the original (x, y, z) coordinate. That is, $M = F_1$, , $N = F_2$, and $P = F_3$. Express the first two components of $\boldsymbol{F}$, (M', N') in the new (x', y', z) coordinate in terms of M and N (P remains unchanged). The same formula also works for the normal vector $\boldsymbol{n} = (n_1, n_2, n_3)$ and the tangent vector $\boldsymbol{T} = (T_1, T_2, T_3)$

(g) Show by direct calculation that

$$\nabla \times \boldsymbol{F} \cdot \boldsymbol{n} = \begin{vmatrix} n_1 & n_2 & n_3 \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \begin{vmatrix} n_1' & n_2' & n_3 \\ \partial_{x'} & \partial_{y'} & \partial_z \\ M' & N' & P \end{vmatrix}$$

and

$$\boldsymbol{F} \cdot \boldsymbol{T} = T_1 F_1 + T_2 F_2 + T_3 F_3 = T_1' F_1' + T_2' F_2' + T_3 F_3$$

With the identities above, one can then perform a few successive rotations to transform a triangle lying in $\mathbb{R}^3$ into a triangle in $x - y$ plan, therefore reducing Stoke's Theorem on a triangle to Green's Theorem in $\mathbb{R}^2$. The latter can be easily verified via Fundamental Theorem of Calculus.

11. Section 16.8: Problems 5, 9, 13, 17, 19, 25, 27, 29, 31.