Volume and Surface Area of Revolution

Notation:

a: axial variable.
r: radial variable.
The generating region:

$$\mathcal{R} = \{(a, r),\ \alpha < a < \beta,\ 0 < r_1(a) < r < r_2(a)\}$$

$$= \{(a, r),\ \gamma < r < \delta,\ 0 < a_1(r) < a < a_2(r)\}$$

Volume of revolution:

Method of disks:

$$V = \int_\alpha^\beta \pi\big(r_2^2(a) - r_1^2(a)\big)da$$

Method of cylindrical shells:

$$V = \int_\gamma^\delta 2\pi r\big(a_2(r) - a_1(r)\big)dr$$

Arclength:

$ds = \sqrt{dx^2 + dy^2}$,

$$L = \int_\alpha^\beta \sqrt{1 + (\frac{dy}{dx})^2}dx = \int_\alpha^\beta \sqrt{1 + \big(f'(x)\big)^2}dx, \quad \text{if } y = f(x)$$

or

$$L = \int_\gamma^\delta \sqrt{1 + (\frac{dx}{dy})^2}dy = \int_\gamma^\delta \sqrt{1 + \big(g'(y)\big)^2}dy, \quad \text{if } x = g(y)$$

Surface of revolution:

a: axial variable.
r: radial variable.

case 1:

The generating curve is $r = f(a)$ on $\alpha < a < \beta$:

$$A = \int_\alpha^\beta 2\pi r ds = \int_\alpha^\beta 2\pi f(a)\sqrt{1 + \big(f'(a)\big)^2}da$$

case 2:

The generating curve is $a = g(r)$ on $\gamma < r < \delta$:

$$A = \int_\gamma^\delta 2\pi r ds = \int_\gamma^\delta 2\pi r\sqrt{1 + \big(g'(r)\big)^2}dr$$