Study Guide for Chap 04

1. Classify possible locations of global minimum or maximum for a continuous function on a closed interval.
2. Read and study the difference between Intermediate Value Theorem and Mean Value Theorem.
3. Study the proofs of Rolle's Theorem and Mean Value Theorem.
4. Study how to determine whether a critical point is a local minimum, local maximum, or neither using the first derivative test.
5. Review procedures for sketching a curve. Such as how to determine whether the curve is increasing/decreasing and concave up/down.
6. Review how to determine a indeterminate form using L'Hôpital's Rule. Go through all examples in section 4.5.
7. Find one example to each of the form $\infty - \infty$, $0 \cdot \infty$, 1^∞, 0^0, ∞^0 and study how to find the limit by L'Hôpital's Rule.
8. Find an example of the form $\frac{0}{0}$ that L'Hôpital's Rule does not leads to an answer, but after rewriting $\frac{0}{0} = \frac{\frac{1}{0}}{\frac{1}{0}} = \frac{\infty}{\infty}$, L'Hôpital's Rule can be applied to find the limit.
9. Review standard procedure for applied optimization: determine relevant range of the unknown, find possible locations of maximum/minimum, and how to verify a candidate point is actually a maximum/minimum.
10. Understand how to derive Newton's method. Study examples where Newton's method does not converge or converge to a wrong solution.
11. Understand the meaning of Antiderivatives and how to find them in the simple cases. Study how to solve initial value problems using antiderivatives and how to find the undetermined constant in the antiderivative from the initial vales (section 4.8, problems 91-113).

Remarks on L'Hôpital's Rule

Theorem 1 (Simple version of L'Hôpital's Rule) *Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

proof :

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}=\lim_{x\to a}\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}}=\frac{\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a}\dfrac{g(x)-g(a)}{x-a}}=\frac{f'(a)}{g'(a)}$$

□

Example 1 $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=\lim\limits_{x\to 0}\dfrac{\cos x}{1}=1$

Theorem 2 (Advanced version of L'Hôpital's Rule (Theorem 6, p255)) *Suppose that $f(a)=g(a)=0$ and that f and g are differentiable on $(a-\delta, a+\delta)$. Suppose also that $g'(x)\neq 0$ if $x\neq a$. If*

$$\lim_{x\to a}\frac{f'(x)}{g'(x)}=\begin{cases} L \\ \infty \\ -\infty \end{cases},$$

Then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Hint: Theorem 2 can be derived easily (at least when the above limit is L), given the following

Theorem 3 *Cauchy's Mean Value Theorem Suppose f and g are continuous on $[a,b]$ and differentiable on (a,b), then there exists $c\in(a,b)$ such that*

$$\begin{vmatrix} f(b)-f(a) & f'(c) \\ g(b)-g(a) & g'(c) \end{vmatrix}=0.$$

Hint: Apply standard Mean Value Theorem to

$$F(x)=\begin{vmatrix} f(b)-f(a) & f(x)-f(a) \\ g(b)-g(a) & g(x)-g(a) \end{vmatrix} \qquad \text{on } [a,b].$$

Example 2 $\lim_{x\to 0}\frac{x-\sin x}{x^3}=$

Remark 1 *Under the same assumption above, if $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ does not exist, it does NOT imply that*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\textit{ non-existent }.$$

Instead, L'Hôpital's Rule gives no conclusion in this case.

Example 3 $\lim_{x\to 0} \frac{x^2 \cos \frac{1}{x}}{\sin x} =$

Hint: L'Hôpital's Rule in inconclusive, use sandwich Theorem instead.

Variants of L'Hôpital's Rule

1. The one sided limit version.
2. The $\frac{\infty}{\infty}$ version.
3. The $\lim_{x\to\infty}$ versions.

In short, whenever you have a indefinite ratio of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, you can simply differentiate both the denominator and the enumerator until you get a limit (either finite or infinite).

The proof for these variants are beyond the scope of this course. You can look at the supplement document 'l'Hôpital.pdf' if you are really curious about it.

Indefinite Differences and Products: $\infty - \infty$ and $0 \cdot \infty$

Example 4 *1.* $\lim\limits_{x\to 0^+} (\frac{1}{\sin x} - \frac{1}{x}) =$

2. $\lim\limits_{x\to\infty} x - \sqrt{x^2 + x} =$

Remark 2 *The choice of writing* $0 \cdot \infty$ *as* $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$ *often makes a technical difference, as the following example shows:*

Example 5

$$\lim_{x\to 0^+} \frac{1}{x} \cdot e^{\frac{-1}{x}} = \lim_{x\to 0^+} \frac{e^{\frac{-1}{x}}}{x} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{e^{\frac{1}{x}}}$$

Which one is better?

Intermediate Powers 1^∞, 0^0 and ∞^0

Example 6 *1.* $\lim_{x\to 0^+} x^{\frac{1}{x}} =$

2. $\lim_{x\to\infty} x^{\frac{1}{x}} =$

3. $\lim_{x\to 0^+} x^x =$

4. $\lim_{x\to 0^+} (1 + ax)^{\frac{b}{x}} =$

Hint: always use the trick $x^y = (e^{\ln x})^y = (e^{y \ln x})$ and continuity of the exponential function: $e^{\lim f(x)} = \lim e^{f(x)}$.