Study Guide for Chap 03

1. Study product rule and applications as higher derivative of product, multiple products, such as
$$\frac{d^n}{dx^n}\big(f(x)g(x)\big) = ?, \qquad \frac{d}{dx}\big(f_1(x)f_2(x)\cdots f_n(x)\big) = ?$$
and determinants, etc.

2. Practice on derivative of composite functions: $\frac{d}{dx}f_1(f_2(\cdots f_n(x)))$ where the functions $f_1, f_2, \cdots f_n$ are elementary functions such as polynomials, trigonometric functions, exponential functions or their combinations.

3. Study how to derive for derivative of inverse functions.

4. Study how to find derivative of a function through implicit differentiation. Also study higher order derivatives. In other words, how to find $\frac{d^n}{dx^n}f(x)$, when the function $y = f(x)$ is defined implicitly through $\{(x, y)|F(x, y) = 0\}$ with a given function F.

5. Study how to find approximate value of functions using linear approximation (linearization) such as $(1 + x)^k$ and $(1 + x)/(1 + y)$, etc. and how to estimate the error of the linear approximation.

6. Study the relation between
$$\lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} = f'(a)$$
and equation (1) on page 207.

7. Study the proof of Chain rule in section 3.11.

Some remarks on inverse functions with new notations

The purpose of this document is to derive the formula for derivative of inverse functions. Instead of the notation (from the textbook)

$$\text{original function: } y = f(x), \qquad \text{inverse function: } y = f^{-1}(x),$$

we shall use the following notations

$$\text{original function: } y = f(x), \qquad \text{inverse function: } \underline{x = f^{-1}(y)},$$

which is, in my opining, the better one.

Inverse Function of $y = f(x)$

A necessary and sufficient condition for

$$f : D_f \longmapsto R_f \qquad (f \text{ maps from domain of } f \text{ to range of } f)$$

to have an inverse function is

"f is one-to-one and onto from domain of f to range of f"

If this is the case, we can define the inverse function

$$f^{-1} : R_f \longmapsto D_f \qquad (f^{-1} \text{ maps from range of } f \text{ to domain of } f)$$

Proposition 1 *If the inverse functions of f exists, then*

- $f^{-1}(f(x)) = x$, *for all* $x \in D_f$.
- $f(f^{-1}(y)) = y$, *for all* $y \in R_f$.

Notice that we have deliberately used a different notation (y) for the argument of f^{-1} to avoid possible confusion. This is different from the textbook.

It is better to use different letters (x and y) for elements in D_f and R_f. We will follow this notation through rest of this note.

The inverse function of $y = f(x)$ is thus denoted by $x = f^{-1}(y)$.

The exponential functions are one-to-one and onto from $\mathbb{R}$ to $\mathbb{R}^+$. The inverse function, denote by $\log_a$ maps from $\mathbb{R}^+$ to $\mathbb{R}$. Therefore

Proposition 2 *We have*

$$\log_a(a^x) = x, \text{ for all } x \in \mathbb{R}, \qquad a^{\log_a y} = y, \text{ for all } y \in \mathbb{R}^+.$$

In particular,

$$\ln(e^x) = x, \text{ for all } x \in \mathbb{R}, \qquad e^{\ln y} = y, \text{ for all } y \in \mathbb{R}^+.$$

Derivative of Inverse Functions and Logarithmic Functions

Since

$$f^{-1}(f(x)) = x \qquad \text{for all } x \in D_f,$$

we take the $x-$ derivative on both sides and use the chain rule to get

$$\frac{d}{dy} f^{-1}(f(x)) \cdot \frac{df(x)}{dx} = \frac{d}{dx} x = 1$$

In other words,

$$\frac{d}{dy} f^{-1}(y)\Big|_{y=f(x)} \cdot \left(\frac{df(x)}{dx}\right) = \frac{d}{dx} x = 1$$

or

$$\frac{d}{dy} f^{-1}(y)\Big|_{y=f(x)} = \frac{1}{\frac{df(x)}{dx}}$$

For example, if $f(x) = e^x$, then $f^{-1}(y) = \ln y$ and we have

$$\frac{d}{dy} \ln y\Big|_{y=e^x} = \frac{1}{\frac{d}{dx}e^x} = \frac{1}{e^x} = \frac{1}{y} \qquad y > 0.$$

Note that the arguments y (of f^{-1}) and x (of f) are evaluated on different points: one on $f(x)$ and the other on x.

The following is WRONG due to confusion from bad notation:

$$\frac{d}{dx} \ln x = \frac{1}{\frac{d}{dx}e^x} = \frac{1}{e^x} = e^{-x}$$

Example 1 *Let $f(x) = x^3 - 3x^2 - 1, x \geq 2$. Find the value of $\frac{df^{-1}(x)}{dx}$ at $x = -1 = f(3)$.*

Hint: to avoid confusion, it is better to change the problem to "Find the value of $\frac{df^{-1}(y)}{dy}$ at $y = -1 = f(3)$".