Supplement to Lecture 02

The precise definition of $\lim_{x\to c} f(x) = L$ is

For any $\varepsilon > 0$, there exists a (corresponding) $\delta > 0$ such that,

$$0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon. \tag{1}$$

In practice, however, we only need to show that (1) holds for small enough $\varepsilon > 0$. This is convenient in many examples. See for instance, the square root in Example 5 of section 2.3.

In other words, we can use

For any $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 > 0$, there exists a (corresponding) $\delta > 0$ such that,

$$0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon. \qquad (1)$$

The reason is, suppose that (1) holds for all $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 > 0$. Then in particular, it holds for $\varepsilon = \varepsilon_0/2$ and therefore we can find the corresponding δ in (1). Denote this particular δ by δ_0, then the statement (1) reads,

$$0 < |x - c| < \delta_0 \implies |f(x) - L| < \varepsilon_0/2. \tag{2}$$

This implies that (1) not only holds for $\varepsilon \in (0, \varepsilon_0)$, it actually holds for any $\varepsilon > 0$. To see this, we simply take $\delta = \delta_0$ for those $\varepsilon \geq \varepsilon_0$. Then from (2), we have

$$0 < |x - c| < \delta_0 \implies |f(x) - L| < \varepsilon_0/2 < \varepsilon. \tag{3}$$

This shows that taking $\delta = \delta_0$ works for those $\varepsilon \geq \varepsilon_0$.