Thomas' Calculus Early Transcendentals 13ed

Remarks on inverse functions with new notations

The purpose of this document is to derive the formula for derivative of inverse functions. Instead of the notation (from the textbook)

original function: y = f(x), inverse function: $y = f^{-1}(x)$,

we shall use the following notations

original function: y = f(x), inverse function: $x = f^{-1}(y)$,

which is, in my opining, the better one.

Inverse Function of y = f(x)

A necessary and sufficient condition for

$$f: D_f \longmapsto R_f \qquad (f \text{ maps from domain of } f \text{ to range of } f)$$

to have an inverse function is

"f is one-to-one and onto from domain of f to range of f"

If this is the case, we can define the inverse function

 $f^{-1}: R_f \longmapsto D_f \qquad (f^{-1} \text{ maps from range of } f \text{ to domain of } f)$

Proposition 1 If the inverse functions of f exists, then

- $f^{-1}(f(x)) = x$, for all $x \in D_f$.
- $f(f^{-1}(y)) = y$, for all $y \in R_f$.

Notice that we have deliberately used a different notation (y) for the argument of f^{-1} to avoid possible confusion. This is different from the textbook.

It is better to use different letters (x and y) for elements in D_f and R_f . We will follow this notation through rest of this note.

The inverse function of y = f(x) is thus denoted by $x = f^{-1}(y)$.

The exponential functions are one-to-one and onto from \mathbb{R} to \mathbb{R}^+ . The inverse function, denote by \log_a maps from \mathbb{R}^+ to \mathbb{R} . Therefore

Proposition 2 We have

$$\log_a(a^x) = x$$
, for all $x \in \mathbb{R}$, $a^{\log_a y} = y$, for all $y \in \mathbb{R}^+$.

In particular,

$$\ln(e^x) = x$$
, for all $x \in \mathbb{R}$, $e^{\ln y} = y$, for all $y \in \mathbb{R}^+$.

Derivative of Inverse Functions and Logarithmic Functions

Since

$$f^{-1}(f(x)) = x$$
 for all $x \in D_f$,

we take the x- derivative on both sides and use the chain rule to get

$$\frac{d}{dy}f^{-1}(f(x))\cdot\frac{df(x)}{dx} = \frac{d}{dx}x = 1$$

In other words,

$$\frac{d}{dy}f^{-1}(y)\Big|_{y=f(x)} \cdot \left(\frac{df(x)}{dx}\right) = \frac{d}{dx}x = 1$$

or

$$\left. \frac{d}{dy} f^{-1}(y) \right|_{y=f(x)} = \frac{1}{\frac{df(x)}{dx}}$$

For example, if $f(x) = e^x$, then $f^{-1}(y) = \ln y$ and we have

$$\frac{d}{dy}\ln y\Big|_{y=e^x} = \frac{1}{\frac{d}{dx}e^x} = \frac{1}{e^x} = \frac{1}{y} \qquad y > 0.$$

Note that the arguments y (of f^{-1}) and x (of f) are evaluated on different points: one on f(x) and the other on x.

The following is WRONG due to confusion from bad notation:

$$\frac{d}{dx}\ln x = \frac{1}{\frac{d}{dx}e^x} = \frac{1}{e^x} = e^{-x}$$

Example 1 Let $f(x) = x^3 - 3x^2 - 1, x \ge 2$. Find the value of $\frac{df^{-1}(x)}{dx}$ at x = -1 = f(3).

Hint: to avoid confusion, it is better to change the problem to "Find the value of $\frac{df^{-1}(y)}{dy}$ at y = -1 = f(3)".