Study guide for quiz 03

Quiz problems include both the lecture contents and homework problems.

1. Section 2.6: Study definition of the limits in p119, p125, p131 and problem 6, homework 04. Find an example for each and for practice proving with these definitions.

2. Section 3.2: Study the definition of derivative, one-sided derivatives, and relation between differentiability and continuity (Theorem 1 and proof).

3. Section 3.3: Study derivative product rule, derivative quotient rule and applications such as
$$\frac{d^n}{dx^n}\big(f(x)g(x)\big) = ?, \qquad \frac{d}{dx}\big(f_1(x)f_2(x)\cdots f_n(x)\big) = ?$$
and derivative of determinants, etc.

1 Brief answers to selected homework problems

Sec 2.6

61. For any $M > 0$, take $\delta = \left(\frac{1}{M}\right)^{\frac{3}{2}} > 0$ s.t.

(a) if $0 < x < \delta$ then $f(x) \geq \frac{1}{x^{\frac{2}{3}}} > \frac{1}{\delta^{\frac{2}{3}}} = M$

(b) if $0 < -x < \delta$ then $f(x) \geq \frac{1}{x^{\frac{2}{3}}} > \frac{1}{\delta^{\frac{2}{3}}} = M$

(c) if $0 < x-1 < \delta$ then $f(x) \geq \frac{2}{(x-1)^{\frac{2}{3}}} > \frac{2}{\delta^{\frac{2}{3}}} = 2M > M$

(d) if $0 < 1-x < \delta$ then $f(x) \geq \frac{2}{(x-1)^{\frac{2}{3}}} > \frac{2}{\delta^{\frac{2}{3}}} = 2M > M$

85. $\lim_{x\to\infty} \left(\sqrt{x^2+3x} - \sqrt{x^2-2x}\right) = \lim_{x\to\infty} \frac{5x}{\sqrt{x^2+3x} + \sqrt{x^2-2x}}$

$= \lim_{x\to\infty} \frac{5}{\sqrt{1+\frac{3}{x}} + \sqrt{1+\frac{2}{x}}}$

$\left(\lim_{x\to\infty} \frac{1}{x} = 0\right) \quad = \frac{5}{2}$

92. For any $M > 0$, take $\delta = \sqrt{\frac{1}{M}} > 0$,

$0 < |x+5| < \delta \Rightarrow 0 < |x+5|^2 < \delta^2 = \frac{1}{M} \Rightarrow \frac{1}{(x+5)^2} > M$

93. (a) $\lim_{x\to c^-} f(x) = \infty \Leftrightarrow$ For any $M > 0$, there exists a $\delta > 0$, s.t. if $-\delta < x-c < 0$ then $f(x) > M$

(b) $\lim_{x\to c^+} f(x) = -\infty \Leftrightarrow$ For any $-M < 0$, there exists a $\delta > 0$ s.t. if $0 < x-c < \delta$ then $f(x) < -M$

(c) $\lim_{x\to c^-} f(x) = -\infty \Leftrightarrow$ For any $-M < 0$, there exists a $\delta > 0$ s.t. if $-\delta < x-c < 0$ then $f(x) < -M$

Figure 1: Brief answers to selected problems in section 2.6, part 1

95. For any $-M<0$, take $\delta = \frac{1}{M} > 0$ s.t.

if $-\delta < x < 0$ then $\frac{1}{x} < \frac{1}{-\delta} = -M$

97. For any $M>0$, take $\delta = \frac{1}{M} > 0$ s.t.

if $0 < x-2 < \delta$ then $\frac{1}{x-2} > \frac{1}{\delta} = M$

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$\lim_{x\to\infty} f(x) = \infty \Leftrightarrow$ for any $M>0$, there exists a $N>0$ s.t.

if $N<x$ then $f(x) > M$

$\lim_{x\to\infty} f(x) = -\infty \Leftrightarrow$ for any $M>0$, there exists a $N>0$ s.t.

if $N<x$ then $f(x) < -M$

$\lim_{x\to-\infty} f(x) = \infty \Leftrightarrow$ for any $M>0$, there exists a $N>0$ s.t.

if $x < -N$ then $f(x) > M$

$\lim_{x\to-\infty} f(x) = -\infty \Leftrightarrow$ for any $M>0$, there exists a $N>0$ s.t.

if $x<-N$ then $f(x) < -M$

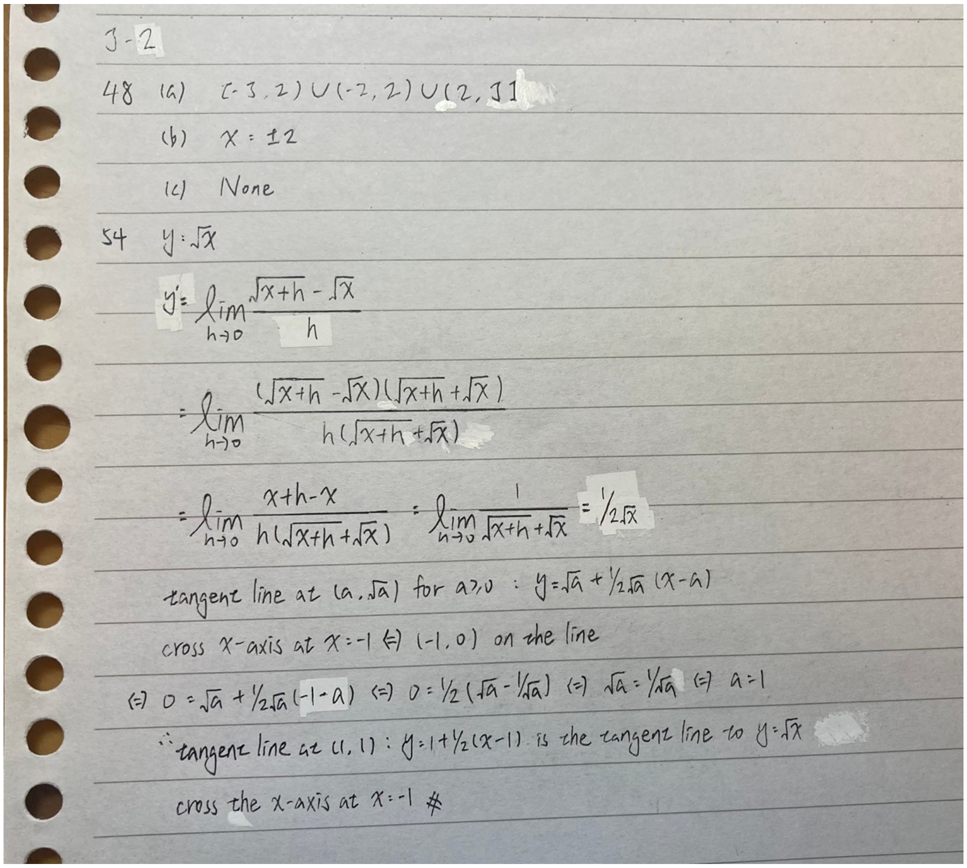
Show: $\lim_{x\to\infty} -x^3 = -\infty$

for any $M>0$, there exists a $N = \sqrt[3]{M} > 0$ s.t.

if $x > N > 0$ then $x^3 > N^3 = M$

$\Rightarrow -x^3 < -M$

Figure 2: Brief answers to selected problems in section 2.6, part 2



3-2

48 (a) $[-3,2)\cup(-2,2)\cup(2,3]$

(b) $x = \pm 2$

(c) None

54 $y = \sqrt{x}$

$$y' = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= \lim_{h\to 0}\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

tangent line at $(a,\sqrt{a})$ for $a \geq 0$: $y = \sqrt{a} + \frac{1}{2\sqrt{a}}(x-a)$

cross x-axis at $x=-1 \Leftrightarrow (-1,0)$ on the line

$\Leftrightarrow 0 = \sqrt{a} + \frac{1}{2\sqrt{a}}(-1-a) \Leftrightarrow 0 = \frac{1}{2}(\sqrt{a} - \frac{1}{\sqrt{a}}) \Leftrightarrow \sqrt{a} = \frac{1}{\sqrt{a}} \Leftrightarrow a = 1$

$\therefore$ tangent line at $(1,1)$: $y = 1 + \frac{1}{2}(x-1)$ is the tangent line to $y=\sqrt{x}$

cross the x-axis at $x=-1$ #

Figure 3: Brief answers to selected problems in section 3.2

4. $\frac{d}{dx}\begin{vmatrix} f_{11}(x) & f_{12}(x) \\ f_{21}(x) & f_{22}(x) \end{vmatrix}$

$= \frac{d}{dx}\left(f_{11}(x) f_{22}(x) - f_{12}(x) f_{21}(x) \right)$

$= f_{11}'(x) f_{22}(x) + f_{11}(x) f_{22}'(x) - f_{12}'(x) f_{21}(x) - f_{12}(x) f_{21}'(x)$

$= \begin{vmatrix} f_{11}'(x) & f_{12}'(x) \\ f_{21}(x) & f_{22}(x) \end{vmatrix} + \begin{vmatrix} f_{11}(x) & f_{12}(x) \\ f_{21}(x)' & f_{22}'(x) \end{vmatrix}$

$= \begin{vmatrix} f_{11}'(x) & f_{12}(x) \\ f_{21}'(x) & f_{22}(x) \end{vmatrix} + \begin{vmatrix} f_{11}(x) & f_{12}'(x) \\ f_{21}(x) & f_{22}'(x) \end{vmatrix}$

5. $\frac{d}{dx}(u(x)v(x)) = uv' + u'v$

$\frac{d^2}{dx^2}(u(x)v(x)) = uv'' + u'v' + u'v' + u''v = uv'' + 2u'v' + u''v$

$\frac{d^3}{dx^3}(u(x)v(x)) = uv''' + 3u'v'' + 3u''v' + u'''v$

Claim $\frac{d^n}{dx^n} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}$

$n=2$ hold, Assume $n=m$ hold

For $n=m+1$, $\frac{d^{m+1}}{dx^{m+1}}(u(x)v(x)) = \frac{d}{dx}\left(\frac{d^m}{dx^m}(u(x)v(x))\right)$

$= \frac{d}{dx}\left(\sum_{k=0}^{m} \binom{m}{k} u^{(k)} v^{(m-k)}\right)$

$= \sum_{k=0}^{m} \binom{m}{k} u^{(k)} v^{(m-k+1)} + \sum_{k=0}^{m} \binom{m}{k} u^{(k+1)} v^{(m-k)} = uv^{(m+1)} + \sum_{k=1}^{m} \underbrace{\left[\binom{m}{k} + \binom{m}{k-1}\right]}_{=\binom{m+1}{k}} u^{(k)} v^{(m-k+1)} + u^{(m+1)}v$

$= \sum_{k=0}^{m+1} \binom{m+1}{k} u^{(k)} v^{(m+1-k)}$ hold by Induction, Claim hold ※

Figure 4: Brief answers to selected problems in section 3.3