Study guide for quiz 01

Quiz problems include both the lecture contents and homework problems.

1. Review Limit laws, technique of rationalizing numerator or denominator to find the limit of the form $\frac{0}{0}$. Review statement and applications of Sandwich Theorem. Keep in mind typical examples of '$\lim_{x \to c} f(x)$ does not exist'.

2. Memorize the precise definition of $\lim_{x \to c} = L$ using ε and δ.

3. Study how to prove $\lim_{x \to c} f(x) = L$ using standard tricks such as the $\epsilon/2$ argument.

1 Some common mistakes in section 2.3, problem 43

2.3 (43) $\lim_{x \to 1} \frac{1}{x} = 1$

For any $\varepsilon > 0$ $1 > \varepsilon > 0$

$\left|\frac{1}{x} - 1\right| < \varepsilon$

$\iff -\varepsilon < \frac{1}{x} - 1 < \varepsilon$

$\iff 1 - \varepsilon < \frac{1}{x} < 1 + \varepsilon$

$\iff \frac{1}{1-\varepsilon} > x > \frac{1}{1+\varepsilon}$

$\iff \frac{1}{1-\varepsilon} - 1 > x - 1 > \frac{1}{1+\varepsilon} - 1$

$\Rightarrow \delta = \min\left\{ \frac{1}{1+\varepsilon} - 1 \text{ X},\ \frac{1}{1-\varepsilon} - 1 \right\}$

take $\delta = \frac{1}{1+\varepsilon} - 1$

$= \frac{1 - 1 - \varepsilon}{1+\varepsilon}$

$= \frac{-\varepsilon}{1+\varepsilon}$

43.

$$0<|x-1|<\delta$$

$$\Rightarrow -\delta<x-1<\delta$$

$$\Rightarrow -\delta+1<x<\delta+1$$

$$\left|\frac{1}{x}-1\right|<\varepsilon \qquad \text{(A)}$$

$$\Leftrightarrow -\varepsilon<\frac{1}{x}-1<\varepsilon$$

$$\Leftrightarrow -\varepsilon+1<\frac{1}{x}<\varepsilon+1 \quad \text{assume } 0<\varepsilon<1$$

$$\Leftrightarrow \frac{1}{1+\varepsilon}<x<\frac{1}{1-\varepsilon} \qquad \text{(B)}$$

$$\begin{cases} \delta+1\le\frac{1}{1-\varepsilon} \\ -\delta+1\ge\frac{1}{1+\varepsilon} \end{cases} \Rightarrow \begin{cases} \delta\le\frac{1}{1-\varepsilon}-1 \\ \delta\le 1-\frac{1}{1+\varepsilon} \end{cases}$$

take $\delta=\min\left\{\frac{1}{1-\varepsilon}-1,\ 1-\frac{1}{1+\varepsilon}\right\}$

因為要說明所選取的delta可以從(B)推到(A),反向箭頭是必要的

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43. $\lim_{x \to 1} \frac{1}{x} = 1$

$1 > \varepsilon > 0$

for any $\varepsilon > 0$, there exist $\delta > 0$

$0 < |x-1| < \delta \Rightarrow \left|\frac{1}{x} - 1\right| < \varepsilon$

$\Leftrightarrow -\varepsilon < \frac{1}{x} - 1 < \varepsilon$

$\Leftrightarrow 1-\varepsilon < \frac{1}{x} < 1+\varepsilon$

$\Leftrightarrow \frac{1}{1-\varepsilon} > x > \frac{1}{1+\varepsilon}$

$\Leftrightarrow \frac{1-(1-\varepsilon)}{1-\varepsilon} > x-1 > \frac{1-(1+\varepsilon)}{1+\varepsilon}$

$\Leftrightarrow \frac{-\varepsilon}{1+\varepsilon} < x-1 < \frac{\varepsilon}{1-\varepsilon}$

take $\delta = \min\left(\frac{\cancel{-}\varepsilon}{1+\varepsilon}, \frac{\varepsilon}{1\cancel{-}\varepsilon}\right)$

$\Leftrightarrow 0 < |x-1| < \delta \Rightarrow \left|\frac{1}{x} - 1\right| < \varepsilon$

43.

prove $\lim_{x\to 1} \frac{1}{x} = 1$

$0<|x-c|<\delta,\ |f(x)-L|<\varepsilon$

Given $1>\varepsilon>0$, there exists $\delta>0$.

$0<|x-c|<\delta,\ |f(x)-L|<\varepsilon$

$\left|\frac{1}{x}-1\right|<\varepsilon$. (A)

$\Leftarrow -\varepsilon<\frac{1}{x}-1<\varepsilon$

$\Leftarrow -\varepsilon+1<\frac{1}{x}<\varepsilon+1$

$\Leftarrow \frac{1}{-\varepsilon+1}>x>\frac{1}{\varepsilon+1}$ (B)

$-\delta<x-1<\delta$

$-\delta+1<x<\delta+1$

$$\begin{cases} 1-\delta \geq \frac{1}{\varepsilon+1} \\ \delta+1 \leq \frac{1}{1-\varepsilon} \end{cases}$$

$$\begin{cases} \delta \leq 1-\frac{1}{1+\varepsilon} = \frac{\varepsilon}{1+\varepsilon} \\ \delta \leq \frac{1}{1-\varepsilon}-1 = \frac{\varepsilon}{1-\varepsilon} \end{cases}$$

$\therefore \delta = \min\left\{\frac{\varepsilon}{1+\varepsilon}, \frac{\varepsilon}{1-\varepsilon}\right\}$

$= \frac{\varepsilon}{1-\varepsilon}$

因為要說明所選取的delta可以從(B)推到(A), 反向的箭頭是必須的

2 Problem 2 of homework week 03

2.1 Correct answer

2. If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$

For any $\varepsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$, such that

$$0 < |x - c| < \delta_1 \Rightarrow |f(x) - L| < \frac{\varepsilon}{6}$$
$$0 < |x - c| < \delta_2 \Rightarrow |g(x) - M| < \frac{\varepsilon}{6}$$
$$\Leftrightarrow \begin{cases} -\frac{2}{3}\varepsilon < 4f(x) - 4L < \frac{2}{3}\varepsilon \\ -\frac{1}{3}\varepsilon < -2g(x) + 2M < \frac{1}{3}\varepsilon \end{cases}$$

take $\delta = \min(\delta_1, \delta_2)$

$$0 < |x - c| < \delta \Rightarrow -\varepsilon < 4f(x) - 2g(x) - (4L - 2M) < \varepsilon$$
$$\Leftrightarrow |4f(x) - 2g(x) - (4L - 2M)| < \varepsilon$$
$$\Rightarrow \lim_{x \to c} (4f(x) - 2g(x)) = 4L - 2M$$

2.2 common mistakes

The most common mistake is to start from

$$a < p < A, \qquad b < q < B.$$

to conclude that

$$a - b < p - q < A - B \quad \text{(Wrong!)}$$

The correct number you should use in your proof is $4 + 2 = 6$, not $4 - 2 = 2$.

The second common mistake is to skip the derivation. You need to first 'break the absolute value', then do the correct addition or subtraction. Skipping the derivation can lead to wrong conclusion, and will lead to point deduction even if the conclusion happens to be correct.

$\boxed{\text{HW W3 二、}}$ If $\lim_{x\to c} f(x)=L$, $\lim_{x\to c} g(x)=M$.

For any $\varepsilon>0$, there exists $\delta_1>0$ and $\delta_2>0$, such that

$0<|x-c|<\delta_1 \Rightarrow -\frac{\varepsilon}{2}<f(x)-L<\frac{\varepsilon}{2}$

$0<|x-c|<\delta_2 \Rightarrow -\frac{\varepsilon}{2}<g(x)-M<\frac{\varepsilon}{2}$

$\Rightarrow$ take $\delta=\min(\delta_1,\delta_2)$

$$\Rightarrow \begin{cases} -2\varepsilon<4f(x)-4L<2\varepsilon \\ -\varepsilon<2g(x)-2M<\varepsilon \end{cases} \cancel{\Rightarrow} -\varepsilon<4f(x)-2g(x)-4L+2M<\varepsilon$$

$\Rightarrow -\varepsilon<4f(x)-2g(x)-(4L-2M)<\varepsilon$

$\Rightarrow |4f(x)-2g(x)-(4L-2M)|<\varepsilon \Rightarrow \lim_{m\to c}(4f(x)-2g(x))=4L-2M.$

$\varepsilon>-2g(x)+2M>-\varepsilon$

$\Rightarrow -2\varepsilon-\varepsilon<4f(x)-2g(x)-(4L-2M)<2\varepsilon+\varepsilon$

二、 For any $\varepsilon>0$, there exist $\delta_1>0$ and δ_2 such that

$0<|x-c|<\delta_1 \Rightarrow -\varepsilon<f(x)-L<\varepsilon \Rightarrow -\frac{\varepsilon}{2}<f(x)-L<\frac{\varepsilon}{2}$ $\times 4$

$0<|x-c|<\delta_2 \Rightarrow -\varepsilon<g(x)-M<\varepsilon \Rightarrow -\frac{\varepsilon}{2}<g(x)-M<\frac{\varepsilon}{2}$ $\times 2$ $)(-$

take $\delta=\min(\delta_1,\delta_2)$

$0<|x-c|<\delta \cancel{\Rightarrow} -\varepsilon<(4f(x)-4L)-(2g(x)-2M)<\varepsilon$ (?)

$\Rightarrow |4f(x)-2g(x)-4L+2M|<\varepsilon$ #

$\lim_{x \to c} f(x) = L$

$0 < |x-c| < \delta_1 \to |f(x) - L| < \frac{\varepsilon}{2}$

$\lim g(x) = M$

$0 < |x-c|\ \delta_2 \to |g(x) - M| < \frac{\varepsilon}{2}$

why

Let $\delta = \min(\delta_1, \delta_2)$

$|4f(x) - 2g(x) - (4L - 2M)| < \varepsilon$

$(c, c+\delta)$

得證

$\delta > 0 \quad x \in (c-\delta, c) \cup$

第二題

② $0 < |x-c| < \delta_1 \Rightarrow |4f(x) - 4L| < \frac{\varepsilon}{2}$ (*)

$0 < |x-c| < \delta_2 \Rightarrow |2g(x) + 2M| < \frac{\varepsilon}{2}$

Why? $\delta \min\{\delta_1, \delta_2\}$

We only have $\lim_{x \to c} f(x) = L$, not $\lim_{x \to c} 4f(x) = 4L$

$-\varepsilon < 4f(x) - 2g(x) - (4L - 2M) < \varepsilon$

$\lim_{x \to c} (4f(x) - 2g(x)) = 4L - 2M$

最好在(*)處加入|f(x)-L| <e/8,這樣就就知道你用的是lim f(x) =L,而非lim 4f(x)=4L (因為後者是需要證明的)