Brief solutions to Quiz 4

Nov 16, 2021

1. (24 + 16 pts) Write down domains and ranges of all six inverse trigonometric functions. Derive the formula for $(\cos^{-1})'$ (pay attention to the $'+'$ or $'-'$ sign in your formula and explain).

   Ans:

   Domains and ranges: See page 3 of Lecture 11 note, or page 62 and page 201 of the textbook (24 **pts**).

   Let $\theta = \cos^{-1}(x)$,

   $$\frac{d}{dx}\cos^{-1}(x) = \frac{1}{\frac{d}{d\theta}\cos(\theta)} = \frac{1}{-\sin\theta} \quad (\mathbf{4pts})$$

   Since $\sin^2\theta + \cos^2\theta = 1$ and $\theta \in [0, \pi]$, it follows that $\sin\theta \geq 0$ therefore $\sin\theta = +\sqrt{1-x^2}$ (**8 pts**).

   Answer: $\dfrac{d}{dx}\cos^{-1}(x) = \dfrac{-1}{\sqrt{1-x^2}}$ (**4 pts**)

2. (15 + 15 pts) Use the linear approximation formula for $(1+x)^k$ near $x = 0$ to get an approximate value of $\dfrac{1}{\sqrt{0.99}}$ and give an estimate of the error for this approximation (i.e. find out $\big|$ true value $-$ approximation $\big| \leq \cdots$). Need not derive the formula you use.

   Ans:

   $$\frac{1}{\sqrt{0.99}} = (1-0.01)^{\frac{-1}{2}} \approx (1 - 0.01\frac{-1}{2}) = 1.005$$

   $$\frac{1}{\sqrt{0.99}} = f(x) = (1+x)^{\frac{-1}{2}}, \quad x = -0.01$$

   $$\Big|\frac{1}{\sqrt{0.99}} - 1.005\Big| \leq \frac{1}{2}\max_{-0.01 \leq c \leq 0}|f''(c)| \cdot 0.01^2$$

   $$f''(c) = \frac{-1}{2}\cdot\frac{-3}{2}(1+c)^{\frac{-5}{2}}$$

   therefore

   $$\Big|\frac{1}{\sqrt{0.99}} - 1.005\Big| \leq \frac{1}{2}|f''(-0.01)| \cdot 0.01^2 = \frac{3}{8}\cdot 0.99^{\frac{-5}{2}} \cdot 0.01^2$$

3. (15 + 15 pts) Let

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Is f differentiable at $x = 0$? Is f twice differentiable at $x = 0$? Start with definitions of $f'(0)$ and $f''(0)$ and explain.

Ans:

$$f'(0) = \lim_{x \to 0} \frac{x^2 \sin(\frac{1}{x}) - 0}{x - 0} = 0$$

$$f'(x) = 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x}), \quad x \neq 0$$

$$\lim_{x \to 0} f'(x) \quad \text{does not exist}$$

So, f' is not continuous at $x = 0$, therefore not differentiable at $x = 0$ and

$$f''(0) = \lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0} \quad \text{does not exist}$$