Brief solutions to Quiz 3

Oct 26, 2021

1. (a) (16 pts) State precise definition of $\lim_{x\to\infty} f(x) = L$.

   (b) (20 pts) Use it to prove that $\lim_{x\to\infty} \frac{1}{x} = 0$.

   Ans: See page 119 of the textbook.

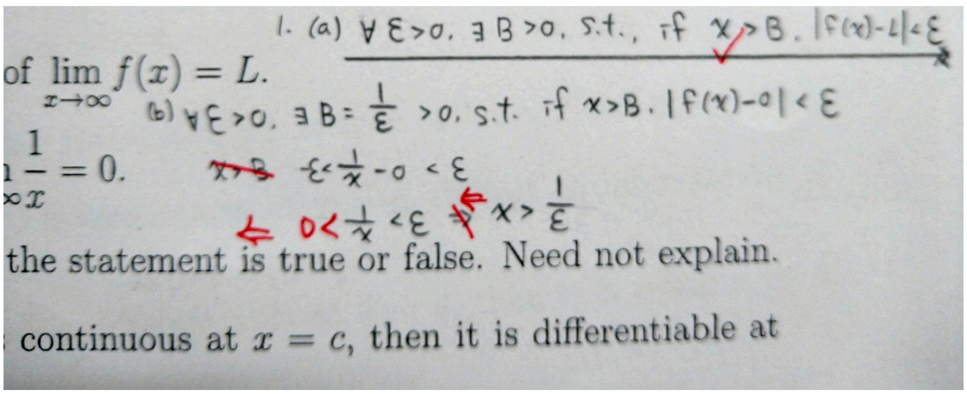

Figure 1: Common mistakes to problem 1 (b)

2. For (a) and (b), just answer whether the statement is true or false. Need not explain.

   (a) (8 pts) True or False? If $f(x)$ is continuous at $x = c$, then it is differentiable at $x = c$.

   (b) (8 pts) True or False? If $f(x)$ is differentiable at $x = c$, then it is continuous at $x = c$.

   (c) (16 pts) Choose either (a) or (b) to elaborate: prove it if true, find a counter example if false.

   Ans: (a): False. (b): True. (either 8 pts or 0 pts, no partial credits)

   (c): If answer (a): counter example: see page 5 of Lecture 05 note. 8 pts for correct example. 8 pts more for correct explanation.

   If answer (b): see Theorem 1 in section 3.2 of the textbook.

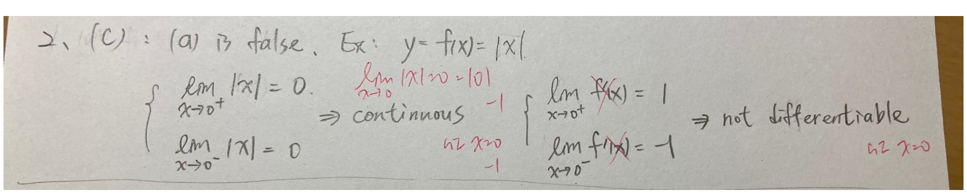

Figure 2: A common mistake to problem 2(c)

Remark on part (c):

Many people chose to answer (a) for part (c) with the counter example $f(x) = |x|$. The true reason for this example to be 'not differentiable at $x = 0$' is

$$\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} \neq \lim_{h\to 0^-} \frac{f(h) - f(0)}{h},$$

and not

$$\lim_{x\to 0^+} f'(x) \neq \lim_{x\to 0^-} f'(x),$$

since $\lim_{h\to 0^+} \frac{f(h) - f(0)}{h}$ and $\lim_{x\to 0^+} f'(x)$ are two different quantities in general. They are the same for the function $f(x) = |x|$ but not always the same. The following is an example with $\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = 0$, while $\lim_{x\to 0^+} f'(x)$ does not exist:

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x > 0, \\ 0, & x = 0. \end{cases}$$

3. Use the derivative product rule to evaluate

$$(a)(16 \text{ pts}) \quad \frac{d^4}{dx^4}\big(f(x)g(x)\big) \qquad (b)(16 \text{ pts}) \quad \frac{d}{dx}\big(f_1(x)f_2(x)f_3(x)f_4(x)\big)$$

Correct answer without derivation receives full credits. If you are sure, just write down the answer directly (and check twice).

Incorrect answer with detailed derivation (starting from $n = 2$, then $n = 3$, then $n = 4$, etc.) will probably receive partial credits if the mistakes are only minor. If you are not sure, try to derive it step by step.

Ans:

(a):

$$f''''(x)g(x) + 4f'''(x)g'(x) + 6f''(x)g''(x) + 4f'(x)g'''(x) + f(x)g''''(x)$$

(b):

$$f_1'(x)f_2(x)f_3(x)f_4(x)+f_1(x)f_2'(x)f_3(x)f_4(x)+f_1(x)f_2(x)f_3'(x)f_4(x)+f_1(x)f_2(x)f_3(x)f_4'(x)$$