Brief solutions to Quiz 1

Oct 12, 2021

1. (20 pts + 20 pts)Evaluate

$$(a) \quad \lim_{x\to 1} \frac{x^{10}-1}{x^3-1} \qquad \text{and} \qquad (b) \quad \lim_{x\to 0} |x|\cos(\frac{1}{x^2})$$

(Need not use the ε-δ argument)

Ans:

(a) Use the identity $a^n - b^n = (a-b)\cdots$. Answer: $\dfrac{10}{3}$.

(b)

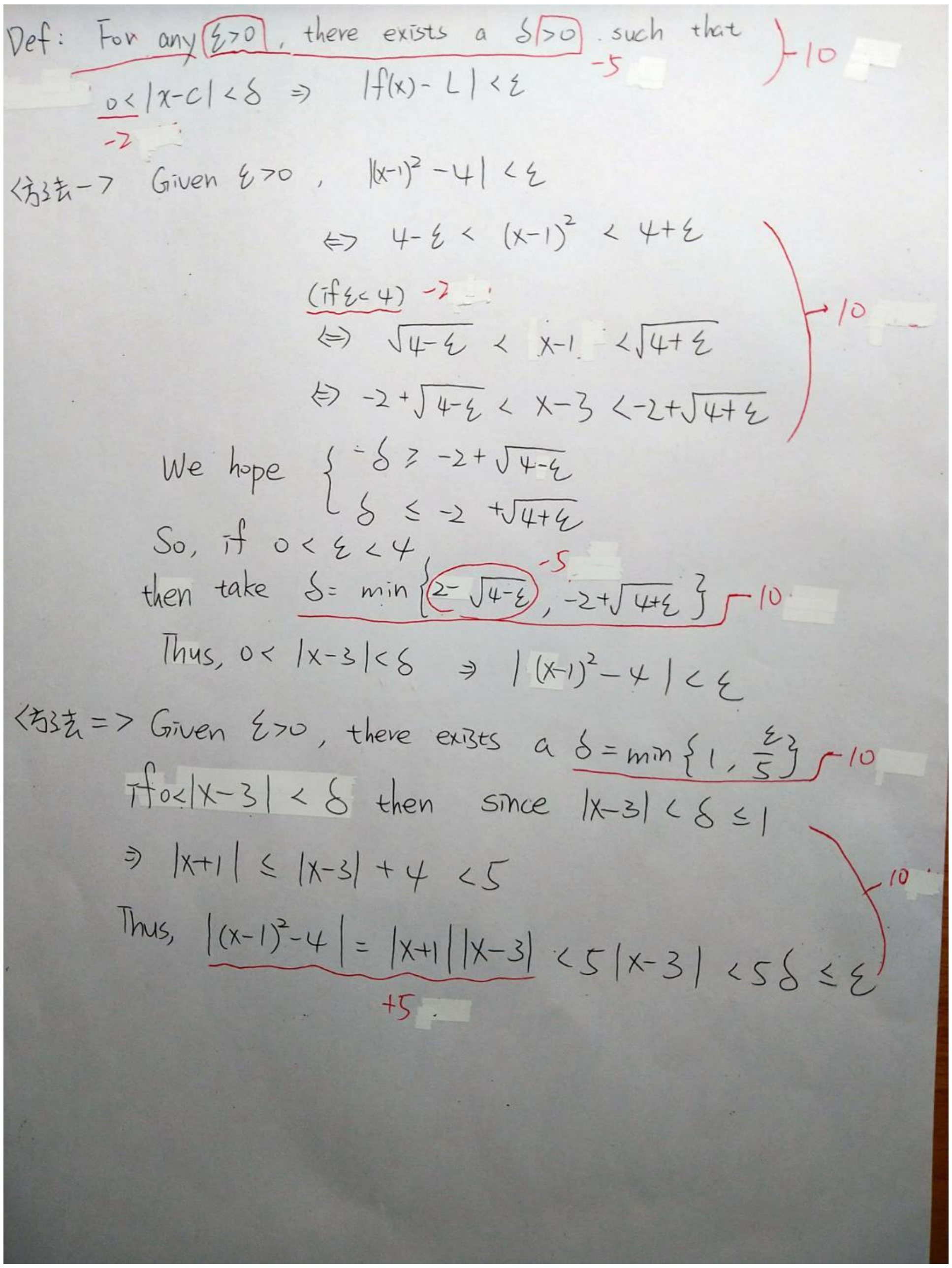
$$-|x| \le |x|\cos(\frac{1}{x^2}) \le |x|$$

Apply the Sandwich Theorem. Answer: 0.

2. (30 pts) Give precise definition of $\lim_{x\to c} f(x) = L$ and use it to show that

$$\lim_{x\to 3}(x-1)^2 = 4$$

Ans:



Def: For any $\boxed{\varepsilon>0}$, there exists a $\delta\boxed{>0}$ such that $0<|x-c|<\delta \Rightarrow |f(x)-L|<\varepsilon$ 10

-5

-2

〈方法一〉 Given $\varepsilon>0$, $|(x-1)^2-4|<\varepsilon$

$\Leftrightarrow 4-\varepsilon < (x-1)^2 < 4+\varepsilon$

(if $\varepsilon<4$) -2

$\Leftrightarrow \sqrt{4-\varepsilon} < x-1 < \sqrt{4+\varepsilon}$ 10

$\Leftrightarrow -2+\sqrt{4-\varepsilon} < x-3 < -2+\sqrt{4+\varepsilon}$

We hope $\begin{cases} -\delta \geq -2+\sqrt{4-\varepsilon} \\ \delta \leq -2+\sqrt{4+\varepsilon} \end{cases}$

So, if $0<\varepsilon<4$,

then take $\delta = \min\{2-\sqrt{4-\varepsilon}, -2+\sqrt{4+\varepsilon}\}$ -5 10

Thus, $0<|x-3|<\delta \Rightarrow |(x-1)^2-4|<\varepsilon$

〈方法二〉 Given $\varepsilon>0$, there exists a $\delta = \min\{1, \frac{\varepsilon}{5}\}$ 10

If $0<|x-3|<\delta$ then since $|x-3|<\delta\leq 1$

$\Rightarrow |x+1| \leq |x-3|+4 < 5$ 10

Thus, $|(x-1)^2-4| = |x+1||x-3| < 5|x-3| < 5\delta \leq \varepsilon$

+5

Figure 1: Answer to problem 2

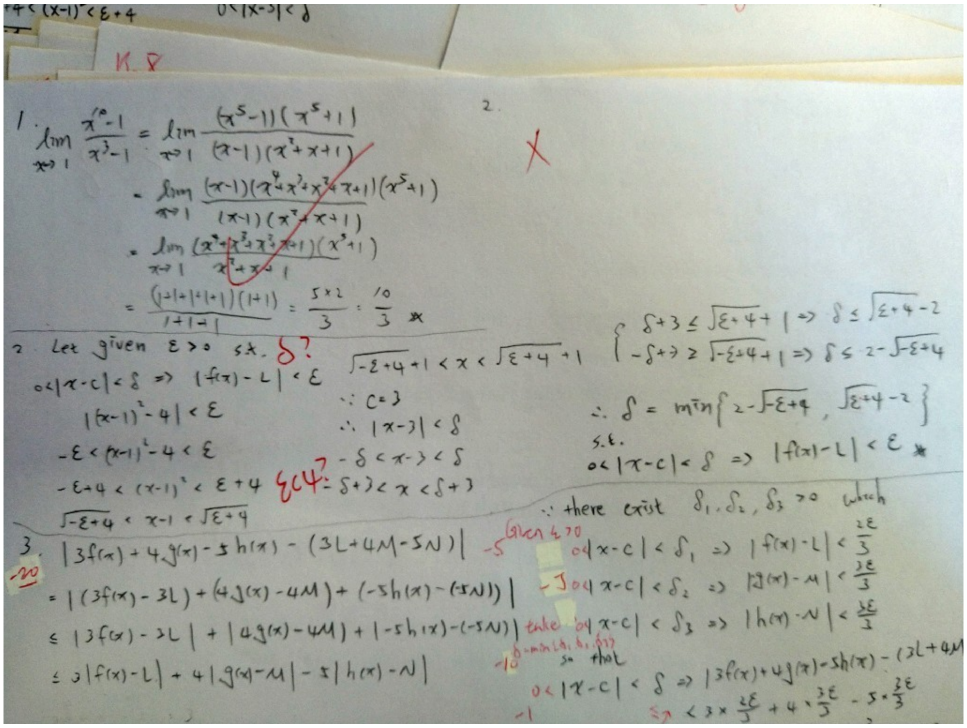


Figure 2: Common mistake (1) of problem 2

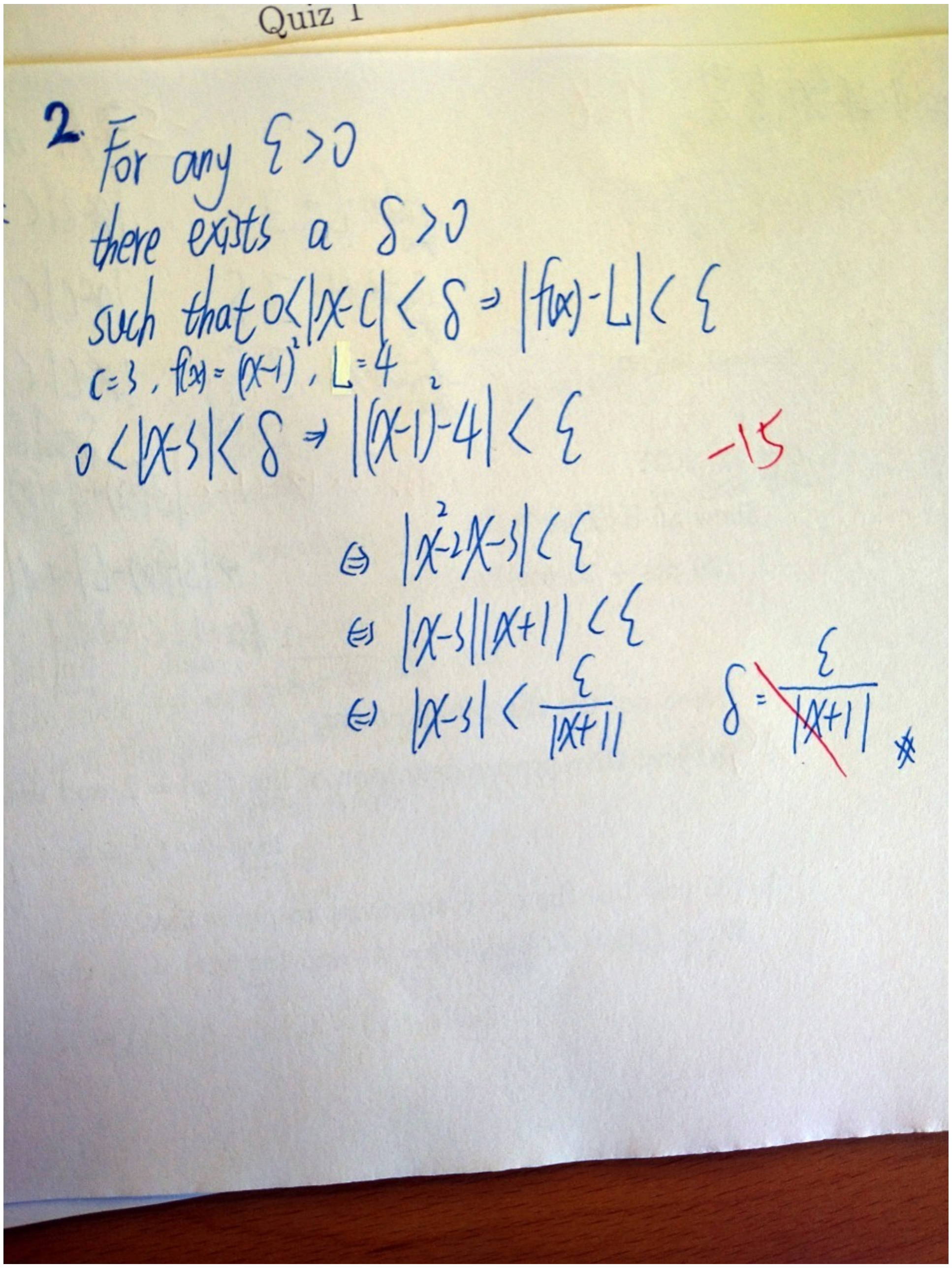
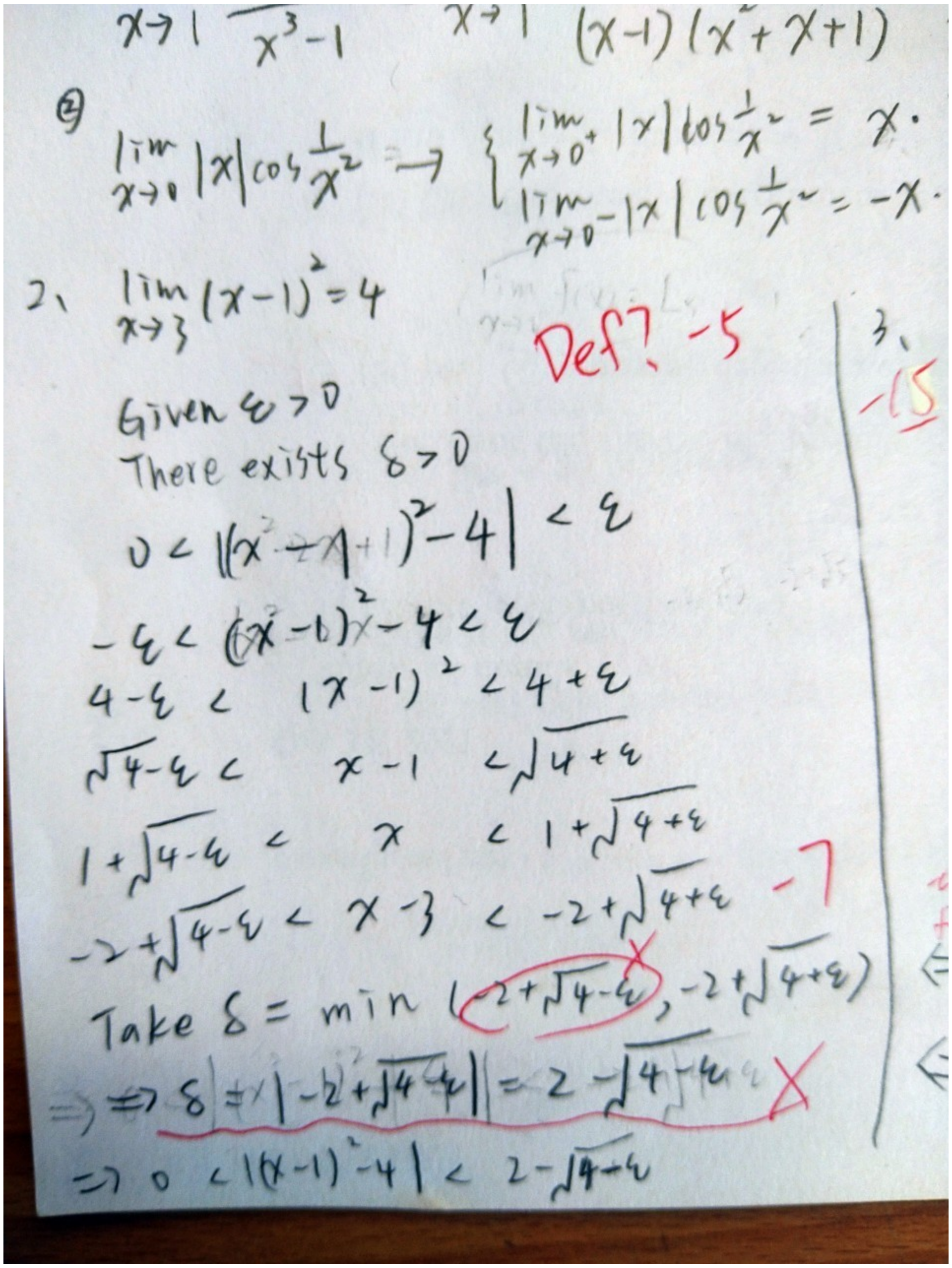


Figure 3: Common mistake (2) of problem 2



$$\lim_{x\to 0}|x|\cos\frac{1}{x^2} \longrightarrow \begin{cases}\lim_{x\to 0^+}|x|\cos\frac{1}{x^2}=x.\\ \lim_{x\to 0^-}-|x|\cos\frac{1}{x^2}=-x.\end{cases}$$

2、 $\lim_{x\to 3}(x-1)^2=4$

Def? -5

Given $\varepsilon>0$

There exists $\delta>0$

$0<|(x-1)^2-4|<\varepsilon$

$-\varepsilon<(x-1)^2-4<\varepsilon$

$4-\varepsilon<(x-1)^2<4+\varepsilon$

$\sqrt{4-\varepsilon}<x-1<\sqrt{4+\varepsilon}$

$1+\sqrt{4-\varepsilon}<x<1+\sqrt{4+\varepsilon}$

$-2+\sqrt{4-\varepsilon}<x-3<-2+\sqrt{4+\varepsilon}$ -7

Take $\delta=\min(-2+\sqrt{4-\varepsilon},-2+\sqrt{4+\varepsilon})$

$\Rightarrow \delta\,|-2+\sqrt{4-\varepsilon}|=2-\sqrt{4-\varepsilon}$

$\Rightarrow 0<|(x-1)^2-4|<2-\sqrt{4-\varepsilon}$

Figure 4: Common mistake (3) of problem 2

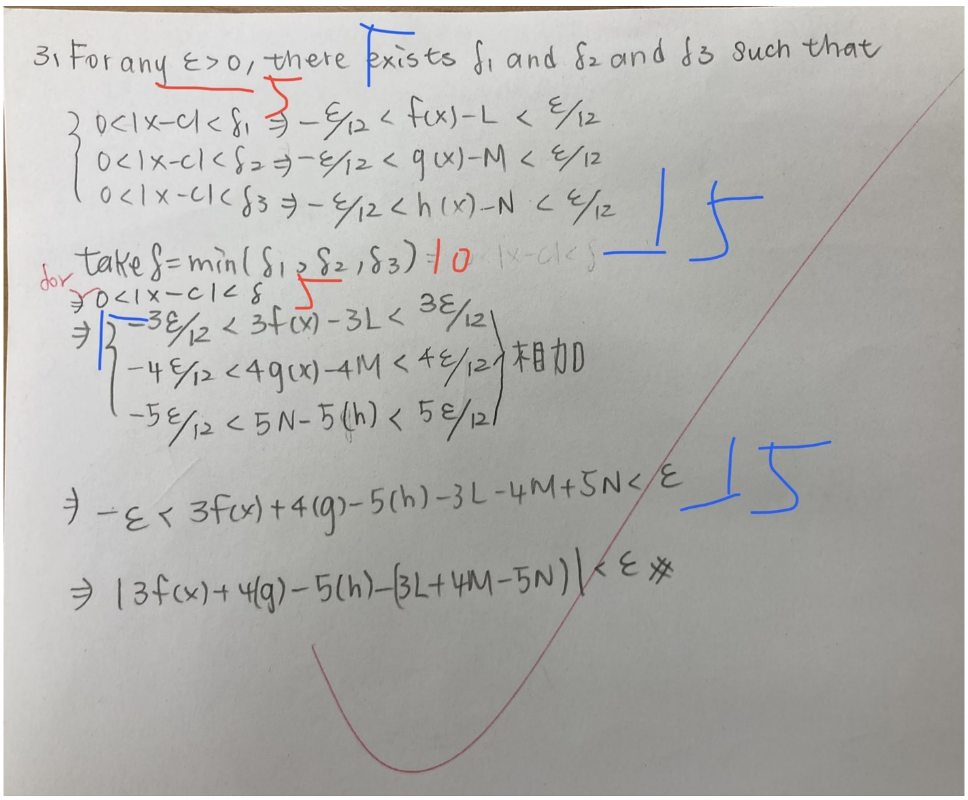
3. (30 pts) Use the $\varepsilon - \delta$ argument to prove that:

If $\lim_{x \to c} f(x) = L$, $\lim_{x \to c} g(x) = M$ and $\lim_{x \to c} h(x) = N$, then

$$\lim_{x \to c} \Big(3f(x) + 4g(x) - 5h(x) \Big) = 3L + 4M - 5N$$

.

Ans:



3, For any $\varepsilon > 0$, there exists δ_1 and δ_2 and δ_3 such that

$$\begin{cases} 0<|x-c|<\delta_1 \ni -\varepsilon/12 < f(x)-L < \varepsilon/12 \\ 0<|x-c|<\delta_2 \ni -\varepsilon/12 < g(x)-M < \varepsilon/12 \\ 0<|x-c|<\delta_3 \ni -\varepsilon/12 < h(x)-N < \varepsilon/12 \end{cases}$$

-15

take $\delta = \min(\delta_1, \delta_2, \delta_3)$ -10 $0<|x-c|<\delta$

for $\ni 0<|x-c|<\delta$

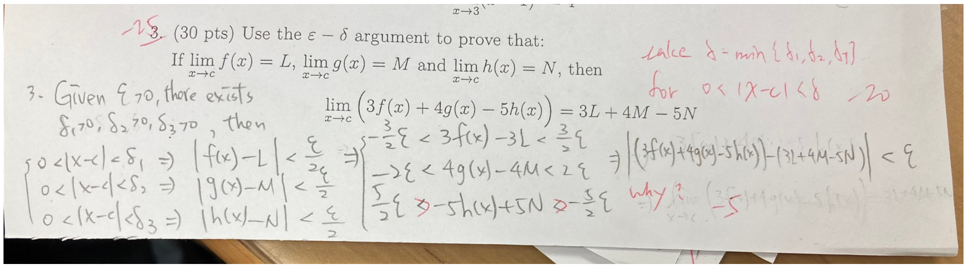
$$\ni \begin{cases} -3\varepsilon/12 < 3f(x) - 3L < 3\varepsilon/12 \\ -4\varepsilon/12 < 4g(x) - 4M < 4\varepsilon/12 \\ -5\varepsilon/12 < 5N - 5(h) < 5\varepsilon/12 \end{cases}$$ 相加

$\ni -\varepsilon < 3f(x) + 4(g) - 5(h) - 3L - 4M + 5N < \varepsilon$ -15

$\ni |3f(x) + 4(g) - 5(h) - (3L + 4M - 5N)| < \varepsilon$ ※

Figure 5: Answer to problem 3

Some common mistakes to problem 3



-25 3. (30 pts) Use the $\varepsilon - \delta$ argument to prove that:

If $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$ and $\lim_{x\to c} h(x) = N$, then

$$\lim_{x\to c}\left(3f(x) + 4g(x) - 5h(x)\right) = 3L + 4M - 5N$$

take $\delta = \min\{\delta_1, \delta_2, \delta_3\}$

for $0 < |x-c| < \delta$ -20

3. Given $\varepsilon > 0$, there exists $\delta_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$, then

$0 < |x-c| < \delta_1 \Rightarrow |f(x) - L| < \frac{\varepsilon}{2}$

$0 < |x-c| < \delta_2 \Rightarrow |g(x) - M| < \frac{\varepsilon}{2}$

$0 < |x-c| < \delta_3 \Rightarrow |h(x) - N| < \frac{\varepsilon}{2}$

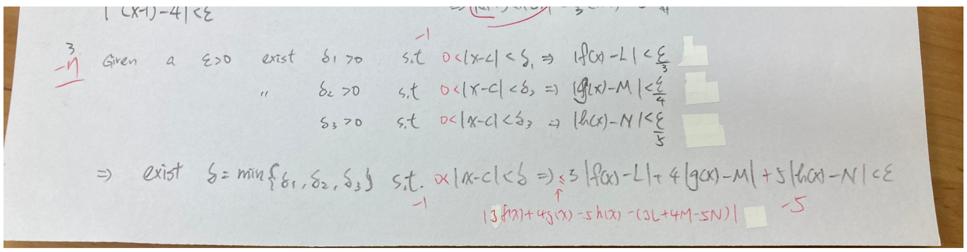
$\Rightarrow -\frac{3}{2}\varepsilon < 3f(x) - 3L < \frac{3}{2}\varepsilon$

$-2\varepsilon < 4g(x) - 4M < 2\varepsilon$

$\frac{5}{2}\varepsilon > -5h(x) + 5N > -\frac{5}{2}\varepsilon$

$\Rightarrow \left|(3f(x)+4g(x)-5h(x)) - (3L+4M-5N)\right| < \varepsilon$ why? -5

Figure 6: A mistake that was already mentioned in guide_mse21f_quiz01.pdf



$|(x-1)-4| < \varepsilon$

3 -17 Given a $\varepsilon > 0$ exist $\delta_1 > 0$ s.t $0 < |x-c| < \delta_1 \Rightarrow |f(x) - L| < \frac{\varepsilon}{3}$ -1

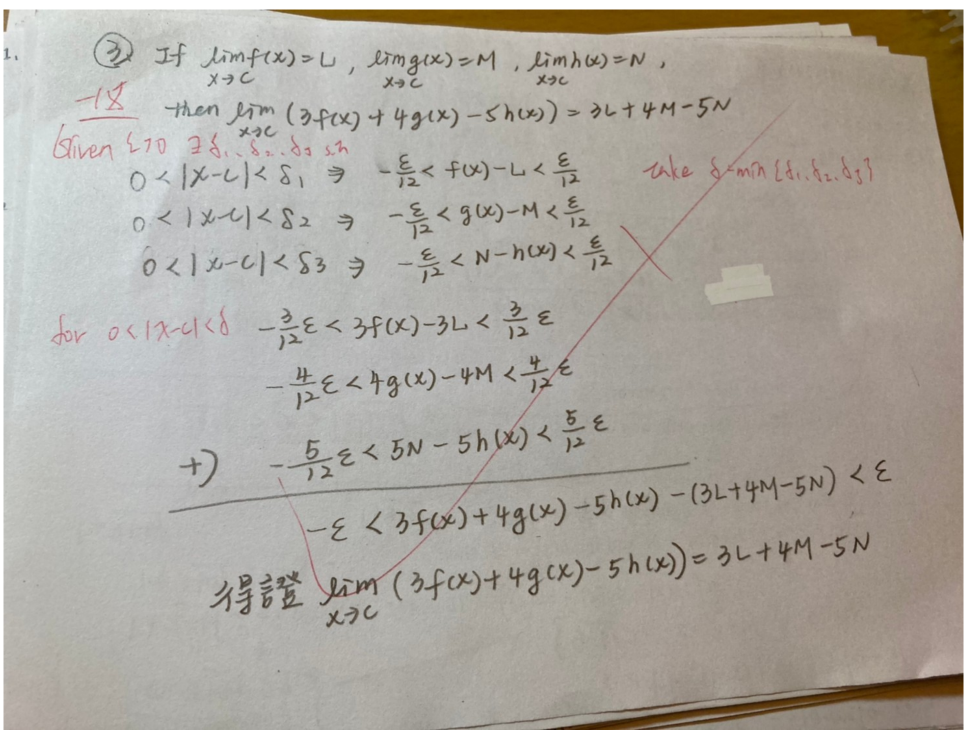
" $\delta_2 > 0$ s.t $0 < |x-c| < \delta_2 \Rightarrow |g(x) - M| < \frac{\varepsilon}{4}$

$\delta_3 > 0$ s.t $0 < |x-c| < \delta_3 \Rightarrow |h(x) - N| < \frac{\varepsilon}{5}$

$\Rightarrow$ exist $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ s.t. $0 < |x-c| < \delta \Rightarrow 3|f(x)-L| + 4|g(x)-M| + 5|h(x)-N| < \varepsilon$ -1

$|3f(x)+4g(x)-5h(x)-(3L+4M-5N)| \le$ -5

Figure 7: Another mistake mentioned in guide_mse21f_quiz01.pdf



1. (3) If $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$, $\lim_{x\to c} h(x) = N$,

-18 then $\lim_{x\to c}(3f(x) + 4g(x) - 5h(x)) = 3L + 4M - 5N$

Given $\varepsilon > 0$ $\exists \delta_1, \delta_2, \delta_3$ s.t.

$0 < |x-c| < \delta_1 \Rightarrow -\frac{\varepsilon}{12} < f(x) - L < \frac{\varepsilon}{12}$ take $\delta = \min\{\delta_1, \delta_2, \delta_3\}$

$0 < |x-c| < \delta_2 \Rightarrow -\frac{\varepsilon}{12} < g(x) - M < \frac{\varepsilon}{12}$

$0 < |x-c| < \delta_3 \Rightarrow -\frac{\varepsilon}{12} < N - h(x) < \frac{\varepsilon}{12}$

for $0 < |x-c| < \delta$ $-\frac{3}{12}\varepsilon < 3f(x) - 3L < \frac{3}{12}\varepsilon$

$-\frac{4}{12}\varepsilon < 4g(x) - 4M < \frac{4}{12}\varepsilon$

$+)\ -\frac{5}{12}\varepsilon < 5N - 5h(x) < \frac{5}{12}\varepsilon$

$-\varepsilon < 3f(x) + 4g(x) - 5h(x) - (3L + 4M - 5N) < \varepsilon$

得證 $\lim_{x\to c}(3f(x) + 4g(x) - 5h(x)) = 3L + 4M - 5N$

Figure 8: Typical mistake to problem 3, incomplete writing

The last one is a common mistake: "What you wrote is not everything you have in mind, resulting an incomplete writing". If you read the last Figure carefully, you will sense that some important sentences are missing (the red ones).

The best way to avoid this kind of mistake from happening again is to read your own writing carefully: line by line, word by word, follow the logic from this line to next line. If it seems OK, then ask your friend, roommate, etc., to read it and see if they understand your writing. Practice a few times, hopefully you will catch the point.