

## Brief solutions to Quiz 1

Oct 12, 2021

1. (20 pts + 20 pts) Evaluate

$$(a) \quad \lim_{x \rightarrow 1} \frac{x^{10} - 1}{x^3 - 1} \quad \text{and} \quad (b) \quad \lim_{x \rightarrow 0} |x| \cos\left(\frac{1}{x^2}\right)$$

(Need not use the  $\varepsilon$ - $\delta$  argument)

**Ans:**

(a) Use the identity  $a^n - b^n = (a - b) \cdots$ . Answer:  $\frac{10}{3}$ .

(b)

$$-|x| \leq |x| \cos\left(\frac{1}{x^2}\right) \leq |x|$$

Apply the Sandwich Theorem. Answer: 0.

2. (30 pts) Give precise definition of  $\lim_{x \rightarrow c} f(x) = L$  and use it to show that

$$\lim_{x \rightarrow 3} (x-1)^2 = 4$$

Ans:

Def: For any  $\epsilon > 0$ , there exists a  $\delta > 0$  such that  
 $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$  -5

方法  $\rightarrow$  Given  $\epsilon > 0$ ,  $|(x-1)^2 - 4| < \epsilon$   
 $\Leftrightarrow 4 - \epsilon < (x-1)^2 < 4 + \epsilon$   
 (if  $\epsilon < 4$ )  $\rightarrow$   
 $\Leftrightarrow \sqrt{4 - \epsilon} < x-1 < \sqrt{4 + \epsilon}$   
 $\Leftrightarrow -2 + \sqrt{4 - \epsilon} < x-3 < -2 + \sqrt{4 + \epsilon}$  10

We hope  $\begin{cases} -\delta \geq -2 + \sqrt{4 - \epsilon} \\ \delta \leq -2 + \sqrt{4 + \epsilon} \end{cases}$   
 So, if  $0 < \epsilon < 4$ ,  
 then take  $\delta = \min \{2 - \sqrt{4 - \epsilon}, -2 + \sqrt{4 + \epsilon}\}$  -5

Thus,  $0 < |x-3| < \delta \Rightarrow |(x-1)^2 - 4| < \epsilon$

方法  $\Rightarrow$  Given  $\epsilon > 0$ , there exists a  $\delta = \min \{1, \frac{\epsilon}{5}\}$  10

If  $0 < |x-3| < \delta$  then since  $|x-3| < \delta \leq 1$   
 $\Rightarrow |x+1| \leq |x-3| + 4 < 5$   
 Thus,  $|(x-1)^2 - 4| = |x+1||x-3| < 5|x-3| < 5\delta \leq \epsilon$  10

+5

Figure 1: Answer to problem 2

3. (30 pts) Use the  $\varepsilon - \delta$  argument to prove that:

If  $\lim_{x \rightarrow c} f(x) = L$ ,  $\lim_{x \rightarrow c} g(x) = M$  and  $\lim_{x \rightarrow c} h(x) = N$ , then

$$\lim_{x \rightarrow c} (3f(x) + 4g(x) - 5h(x)) = 3L + 4M - 5N$$

Ans:

3. For any  $\varepsilon > 0$ , there exists  $\delta_1$  and  $\delta_2$  and  $\delta_3$  such that

$$\begin{cases} 0 < |x - c| < \delta_1 \Rightarrow -\varepsilon/12 < f(x) - L < \varepsilon/12 \\ 0 < |x - c| < \delta_2 \Rightarrow -\varepsilon/12 < g(x) - M < \varepsilon/12 \\ 0 < |x - c| < \delta_3 \Rightarrow -\varepsilon/12 < h(x) - N < \varepsilon/12 \end{cases}$$

take  $\delta = \min(\delta_1, \delta_2, \delta_3) = 0$   $|x - c| < \delta$  15

do  $\Rightarrow 0 < |x - c| < \delta$

$$\begin{cases} -3\varepsilon/12 < 3f(x) - 3L < 3\varepsilon/12 \\ -4\varepsilon/12 < 4g(x) - 4M < 4\varepsilon/12 \\ -5\varepsilon/12 < 5h(x) - 5N < 5\varepsilon/12 \end{cases}$$

相加

$$\Rightarrow -\varepsilon < 3f(x) + 4g(x) - 5h(x) - 3L - 4M + 5N < \varepsilon$$

15

$$\Rightarrow |3f(x) + 4g(x) - 5h(x) - (3L + 4M - 5N)| < \varepsilon$$

Figure 2: Answer to problem 3

### Some common mistakes to problem 3

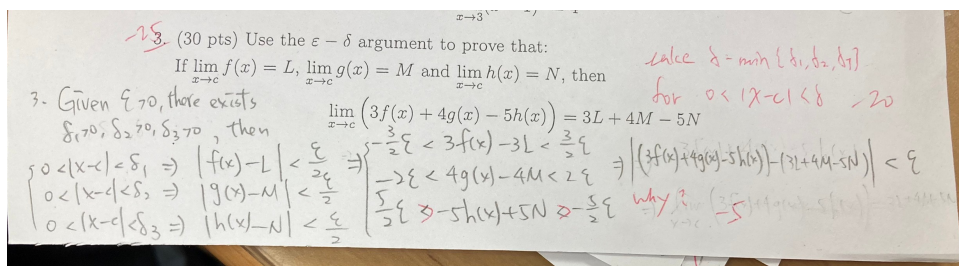


Figure 3: A mistake that was already mentioned in `guide_mse21f_quiz01.pdf`

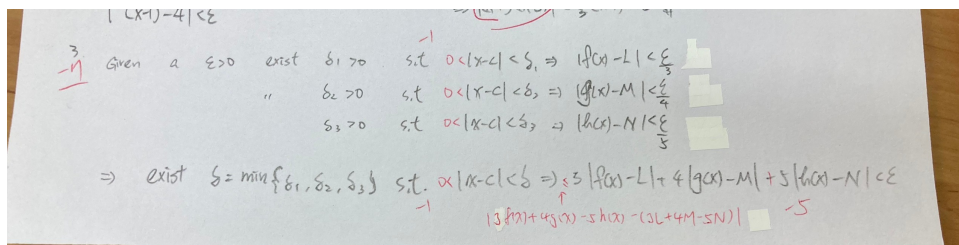


Figure 4: Another mistake mentioned in guide\_mse21f\_quiz01.pdf

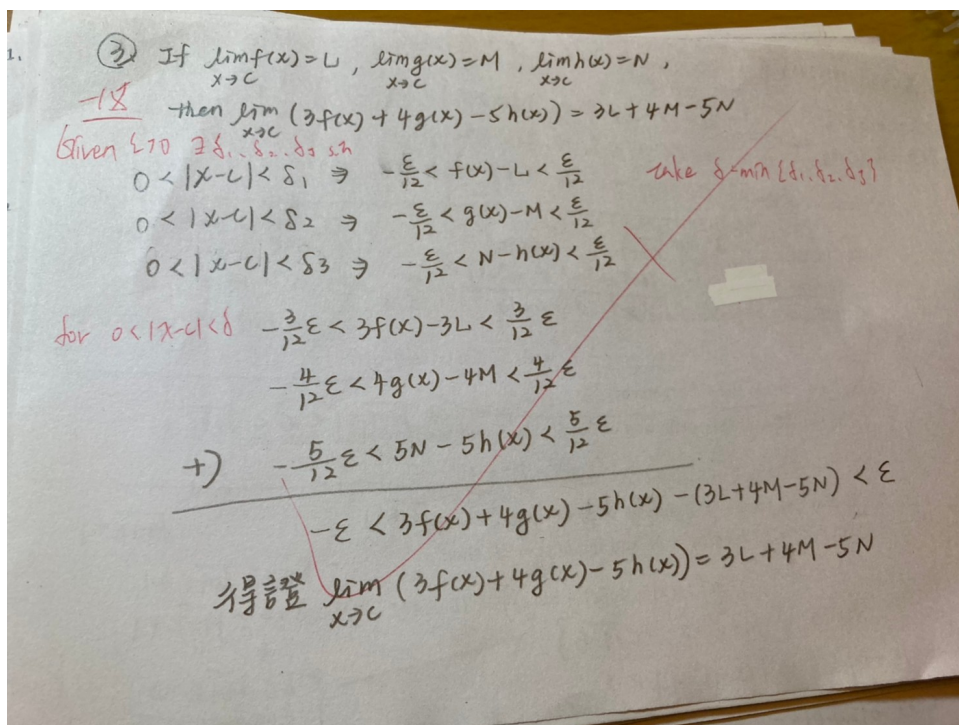


Figure 5: Typical mistake to problem 3, incomplete writing

The last one is a common mistake: "What you wrote is not everything you have in mind, resulting an incomplete writing". If you read the last Figure carefully, you will sense that some important sentences are missing (the red ones).

The best way to avoid this kind of mistake from happening again is to read your own writing carefully: line by line, word by word, follow the logic from this line to next line. If it seems OK, then ask your friend, roommate, etc., to read it and see if they understand your writing. Practice a few times, hopefully you will catch the point.