Brief solutions to Midterm 02

Dec 09, 2021. 10 pts each problem.

1. (5 pts + 5 pts) Write down domains and ranges of all six inverse trigonometric functions. Derive the formula for $(\csc^{-1})'$ (pay attention to the $'+'$ or $'-'$ sign in your formula and explain).

   Ans:

   See quiz 4 solution (please do read).

2. (5 pts + 5 pts) Use the linear approximation to find an approximate value of $\sqrt{1.001}$ and give an estimate of the error for this approximation. Need not derive the formula you use.

   Ans:

   Linear approximation $= 1 + \frac{1}{2}0.001 = 1.0005$.

   Error $\leq |\frac{1}{2}\frac{1}{2}(\frac{-1}{2})0.001^2| = 1.25 \cdot 10^{-7}$.

3. (2 pts + 8 pts) True or False? If true, prove it. If false, give a counter example.

   If $|f(x) - (3x + 2)| \leq |x|^{1.5}$ for all $x \in R$, then f is differentiable at $x = 0$.

   Ans:

   True. Since $\lim_{x\to 0} \dfrac{f(x) - (3x+2)}{|x-0|} = 0$. Therefore $3x + 2$ is the linear approximation of f near $x = 0$ and $f'(0) = 3$.

4. (5 pts + 5 pts) Let f be a differentiable function defined on $\{x \geq 0\}$ satisfying

   (a) $f(0) = -1$, (b) $f'(x) \geq 1/2$ for all $x \geq 0$.

   Show that $f(x) = 0$ has exactly one solution on $\{x \geq 0\}$.

   Ans:

   First, we show that $f(3) > 0$. Otherwise, if $f(3) \leq 0$, then by Mean Value Theorem, there exists a $c \in (0, 3)$ such that $f'(c) = \dfrac{f(3) - f(0)}{3 - 0} \leq \dfrac{0 - (-1)}{3 - 0} = \dfrac{1}{3}$, a contradiction.

   Secondly, since $f(0) < 0$, $f(3) > 0$, by Intermediate Value Theorem, there exists a $c_1 \in (0, 3)$ such that $f(c_1) = 0$.

   Finally, we show that $f(x) = 0$ has only one solution. Suppose not, we will have $c_1 \geq 0$, $c_2 \geq 0$, $c_1 \neq c_2$ such that $f(c_1) = f(c_2) = 0$. From Rolle's Theorem, there exists an $\alpha \in (c_1, c_2)$ such that $f'(\alpha) = 0$, a contradiction.

5. (3 pts + 2 pts + 5 pts) Find all critical points of $f(x) = x^{\frac{1}{3}}(x-4)$ on $[-1,5]$. For each one of them, use first derivative test to determine whether it corresponds to a local minimum, a local maximum or neither. Then find the absolute max and min on $[-1,5]$.

   Ans:

   $f'(x) = \frac{4}{3}x^{\frac{-2}{3}}(x-1)$. Critical points: $x = 0$, $x = 1$.

   $f'(x) < 0$ on $[-1,1)$, $f'(x) > 0$ on $(1,5)$. So $f(1) =$ only local min, therefore absolute min.

   Local max: $f(-1) = 5$, $f(5) = 5^{\frac{1}{3}}$. Compare values of $f(-1)$ and $f(5)$, we conclude that $f(-1) =$ absolute max.

   $x = 0$ is neither local min nor local max.

6. (4 pts + 6 pts) Let

$$f(x) = \begin{cases} e^{(\frac{-1}{x^2})}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

   Is f differentiable at $x = 0$? Is f twice differentiable at $x = 0$? Start with definitions of $f'(0)$ and $f''(0)$ and explain.

   Ans:

   Yes. Yes.

   $f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{\frac{1}{x}}{e^{x^{-2}}}$ (apply L'Hôpital's Rule for $\frac{\pm\infty}{\infty}$ once) $= \lim_{x\to 0} \frac{-x^{-2}}{-2x^{-3}e^{x^{-2}}} = \lim_{x\to 0} \frac{x}{2e^{x^{-2}}} = 0$.

   $f'(x) = \frac{2}{x^3}e^{(\frac{-1}{x^2})}$ for $x \neq 0$.

   $f''(0) = \lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0} \frac{\frac{2}{x^4}}{e^{\frac{1}{x^2}}} = \lim_{x\to 0} \frac{2x^{-4}}{e^{x^{-2}}}$ (apply L'Hôpital's Rule for $\frac{\pm\infty}{\infty}$ twice) $= \lim_{x\to 0} \frac{-8x^{-5}}{-2x^{-3}e^{x^{-2}}} = \lim_{x\to 0} \frac{4x^{-2}}{e^{x^{-2}}} = \lim_{x\to 0} \frac{-8x^{-3}}{-2x^{-3}e^{x^{-2}}} = \lim_{x\to 0} \frac{4}{e^{x^{-2}}} = 0$

7. (4 pts + 6 pts) Evaluate (a) $\lim_{x\to 0^+} x^x$ (b) $\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$

   Ans:

   (a): See quiz 6 solution.

   (b): Let $f(x) = x^2$, $g(x) = \sin^2(x)$ and denote k-th derivative of f, g by $f_k(x) = f^{(k)}(x)$, $g_k(x) = g^{(k)}(x)$.

   We have

$$f_0(x) = x^2, \quad f_1(x) = 2x, \quad f_2(x) = 2, \quad f_3(x) = 0, \quad f_4(x) = 0,$$

$$g_0(x) = \sin^2 x, \quad g_1(x) = \sin 2x, \quad g_2(x) = 2\cos 2x, \quad g_3(x) = -4\sin 2x, \quad g_4(x) = -8\cos 2x$$

$$\lim_{x\to 0}\Big(\frac{1}{x^2}-\frac{1}{\sin^2 x}\Big)=\lim_{x\to 0}\frac{g_0(x)-f_0(x)}{f_0(x)g_0(x)}\ \Big(\text{"}=(\frac{0}{0})\text{"}\Big)=\lim_{x\to 0}\frac{g_1(x)-f_1(x)}{f_0(x)g_1(x)+f_1(x)g_0(x)}\ \Big(\text{"}=(\frac{0}{0})\text{"}\Big)$$

$$=\lim_{x\to 0}\frac{g_2(x)-f_2(x)}{f_0(x)g_2(x)+2f_1(x)g_1(x)+f_2(x)g_0(x)}\ \Big(\text{"}=(\frac{0}{0})\text{"}\Big)$$

$$=\lim_{x\to 0}\frac{g_3(x)-f_3(x)}{f_0(x)g_3(x)+3f_1(x)g_2(x)+3f_2(x)g_1(x)+f_3(x)g_0(x)}\ \Big(\text{"}=(\frac{0}{0})\text{"}\Big)$$

$$=\lim_{x\to 0}\frac{g_4(x)-f_4(x)}{f_0(x)g_4(x)+4f_1(x)g_3(x)+6f_2(x)g_2(x)+4f_3(x)g_1(x)+f_4(x)g_0(x)}=\frac{-8\cos 0}{6\cdot 2\cdot 2\cos 0}=\frac{-1}{3}$$

8. (5 pts + 5 pts) Solve for $y(x)$ on $x<0$ from $\quad y''(x)=x^{-2},\quad y(-1)=1,\quad y'(-1)=2.$

   Ans:

   $$y'(x)=y'(-1)+\int_{-1}^{x}t^{-2}dt=2+(-t^{-1})\Big|_{-1}^{x}=-\frac{1}{x}+1$$

   $$y(x)=y(-1)+\int_{-1}^{x}\big(-\frac{1}{t}+1\big)dt=1+\big(-\ln|t|+t\big)\Big|_{-1}^{x}=-\ln(-x)+x+2$$

9. (2 pts + 2 pts + 6 pts) State both parts of Fundamental Theorem of Calculus, prove that 'part 1 implies part 2'. If you can't prove this, you could prove 'part 1' instead.

   Ans:

   See the textbook or Lecture 19 note.

10. (5 pts + 5 pts) Evaluate (a) $\displaystyle\int_1^2\frac{1}{x(1+\ln^2 x)}dx$ (b) $\displaystyle\int_0^4 x\sqrt{2x+1}\,dx$

    Ans:

    (a): $\displaystyle=\int_1^2\frac{1}{(1+\ln^2 x)}d\ln x=\int_1^2\tan^{-1}(\ln x)=\tan^{-1}(\ln 2)$

    (b): See Section 5.5, Example 6 for the indefinite integral part, then substitute to evaluate the definite integral.

11. (2 pts + 8 pts) True or False? If true, prove it. If false, give a counter example.

    If $y=f(x)$ is differentiable at $x=c$ then it is continuous at $x=c$.

    Ans:

    See midterm 1 solution (same problem, please do read).

12. (average = 4.86 pts) Evaluate $\dfrac{d}{dx}(\tan x)^{\sin x}$, $0<x<\frac{\pi}{2}$.

    Ans:

    See midterm 1 solution (similar (almost the same) problem, please do read).