Brief solutions to Midterm 01

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1. (6 pts) Find $\lim_{y\to+\infty} y\sin\frac{2}{\sqrt{y}}$.

   Ans:

$$\lim_{y\to+\infty} y\sin\frac{2}{\sqrt{y}} = \lim_{y\to+\infty}\sqrt{y}\frac{\sin\frac{2}{\sqrt{y}}}{\frac{1}{\sqrt{y}}} = \lim_{y\to+\infty}\sqrt{y}\cdot\lim_{x\to 0}\frac{\sin(2x)}{x} = \infty\cdot 2 = \infty$$

2. (10 pts) Evaluate $\lim_{z\to x}\frac{\sec(z^2+1)-\sec(x^2+1)}{e^z-e^x}$. Hint: what is the definition of $f'(x)$?

   Ans:

$$\lim_{z\to x}\frac{\sec(z^2+1)-\sec(x^2+1)}{e^z-e^x} = \lim_{z\to x}\frac{\sec(z^2+1)-\sec(x^2+1)}{z-x}\lim_{z\to x}\frac{z-x}{e^z-e^x}$$

$$= \frac{\lim_{z\to x}\frac{\sec(z^2+1)-\sec(x^2+1)}{z-x}}{\lim_{z\to x}\frac{e^z-e^x}{z-x}} = \frac{\frac{d}{dx}\sec(x^2+1)}{\frac{d}{dx}e^x} \quad (\textbf{5pts})$$

$$= \frac{\sec(x^2+1)\cdot\tan(x^2+1)\cdot 2x}{e^x} \quad (\textbf{5pts})$$

3. (4+4+8 pts)

   (a) Write down precise definition of $\lim_{x\to c} f(x) = L$

   (b) State the Sandwich Theorem.

   (c) Prove the Sandwich Theorem using the definition in (a).

   Ans:

   (a),(b): see textbook.

   (c): Suppose $g(x)\le f(x)\le h(x)$ on $(c-a.c)\cup(c,c+a)$ (that is, $\{x \mid 0<|x-c|<a\}$), $a>0$ and $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$. Then for any $\varepsilon>0$, there exists a $\delta_1>0$ and a $\delta_2>0$ such that

$$0<|x-c|<\delta_1 \implies |g(x)-L|<\varepsilon \implies L-\varepsilon<g(x)$$

   and

$$0<|x-c|<\delta_2 \implies |h(x)-L|<\varepsilon \implies h(x)<L+\varepsilon$$

   Take $\delta=\min\{\delta_1,\delta_2,a\}$. It follows that

$$0<|x-c|<\delta \implies L-\varepsilon<g(x)\le f(x)\le h(x)<L+\varepsilon \implies |f(x)-L|<\varepsilon.$$

4. (2+8 pts) True or False? Prove it if true, find a counter example if false.
   If $f(x)$ is differentiable at $x = c$, then it is continuous at $x = c$.

   Ans:

   True. See textbook for the proof.

5. (12 pts) Let $f_i(x) = a_i x^2 + b_i x + c_i$, $i = 1, 2, 3$. Evaluate $\dfrac{d}{dx}\begin{vmatrix} f_1(x) & f_1'(x) & f_1''(x) \\ f_2(x) & f_2'(x) & f_2''(x) \\ f_3(x) & f_3'(x) & f_3''(x) \end{vmatrix}$.

   Ans:

$$= \begin{vmatrix} f_1'(x) & f_1'(x) & f_1''(x) \\ f_2'(x) & f_2'(x) & f_2''(x) \\ f_3'(x) & f_3'(x) & f_3''(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_1''(x) & f_1''(x) \\ f_2(x) & f_2''(x) & f_2''(x) \\ f_3(x) & f_3''(x) & f_3''(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_1'(x) & f_1'''(x) \\ f_2(x) & f_2'(x) & f_2'''(x) \\ f_3(x) & f_3'(x) & f_3'''(x) \end{vmatrix} = 0$$

   The first 2 determinants are zero since they both have identical columns. The third determinant is zero since $f_i'''(x) = 0$.

6. (12 pts) Find the smallest positive integer n such that $\dfrac{d^n}{dx^n}(x^{10}\sin x)\Big|_{x=0}$ is nonzero and find this value. Explain.

   Ans: Since

$$\frac{d^n}{dx^n}(x^{10}\sin x)\Big|_{x=0} = \sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dx^k}x^{10}\Big|_{x=0}\frac{d^{n-k}}{dx^{n-k}}\sin x\Big|_{x=0}$$

   and

$$\frac{d^p}{dx^p}x^{10}\Big|_{x=0} \neq 0 \iff p = 10, \qquad \frac{d^q}{dx^q}\sin x\Big|_{x=0} \neq 0 \iff q = 1, 3, 5, \cdots,$$

   It follows that the smallest $n = 11$ $(p = 10, q = 1)$.

   Answer $= \binom{11}{10} \cdot 10! \cdot \cos 0 = 11!$

7. (10 pts) Evaluate $\dfrac{d}{dx}(\tan x)^{\sqrt{x}}$, $0 < x < \frac{\pi}{2}$.

   Ans:

$$(\tan x)^{\sqrt{x}} = (e^{\ln(\tan x)})^{\sqrt{x}} = e^{\sqrt{x}\ln(\tan x)},$$

   therefore, the answer is (need not simplify)

$$\frac{d}{dx}e^{\sqrt{x}\ln(\tan x)} = (\tan x)^{\sqrt{x}}\frac{d}{dx}(\sqrt{x}\ln(\tan x)) = (\tan x)^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\ln(\tan x) + \sqrt{x}\frac{\sec^2 x}{\tan x}\right)$$

8. (12 pts) Find the normal line and tangent line of $x^4 + y^2 = 2$ at $(1, -1)$.

   Ans:

$$x^4 + y^2 = 2 \implies 4x^3 + 2yy' = 0 \implies y' = -2\frac{x^3}{y} = 2 \quad (4\textbf{pts})$$

$$\text{tangent line}: \quad \frac{y+1}{x-1} = 2, \quad (\textbf{4pts}) \quad \text{normal line}: \quad \frac{y+1}{x-1} = -\frac{1}{2}. \quad (\textbf{4pts})$$

9. (4+8 pts) Let f^{-1} be the inverse function of f. Suppose f and f^{-1} are twice differentiable (i.e. both first and second derivative exist).

(a) <u>Derive</u> the formula of $\frac{d}{dy}f^{-1}(y)$ as a function of y in terms of $f'(\cdot)$ and $f^{-1}(\cdot)$.

(b) <u>Derive</u> the formula of $\frac{d^2}{dy^2}f^{-1}(y)$ as a function of y in terms of $f'(\cdot)$, $f''(\cdot)$ and $f^{-1}(\cdot)$.

Ans:

Denote by $y = f(x)$, or $x = f^{-1}(y)$.

(a):

$$f^{-1}(f(x)) = x \implies (f^{-1})'(y) \cdot f'(x) = 1 \implies (f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))}$$

(b):

$$\frac{d}{dx}\Big((f^{-1})'(y) \cdot f'(x) = 1\Big) \implies (f^{-1})''(y) \cdot (f'(x))^2 + (f^{-1})'(y) \cdot f''(x) = 0$$

$$\implies (f^{-1})''(y) = \frac{-(f^{-1})'(y) \cdot f''(x)}{(f'(x))^2} = \frac{-f''(x)}{(f'(x))^3} = \frac{-f''\big(f^{-1}(y)\big)}{\Big(f'\big(f^{-1}(y)\big)\Big)^3}$$

Method 2 for (b): take $\frac{d}{dx}$ on (a) and use quotient rule and the Chain Rule.