Brief solutions to Final Exam

Jan 13, 2022.

1. (12 pts) Find the volume and surface area of the object obtained by rotating the region $\{(x-2)^2 + y^2 \le 1,\ x \ge 2\}$ around the y axis. Note the surface area consists of two parts, one generated by a half circle, the other generated by a line segment.

   Ans:

   Volume $= \int_{-1}^{1} \pi((2+\sqrt{1-y^2})^2 - 2^2)dy = 2\pi^2 + \frac{4}{3}\pi$.

   Here we have used $\int_{-1}^{1}\sqrt{1-y^2}\,dy = \int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta = \int_{-\pi/2}^{\pi/2}\frac{\cos 2\theta+1}{2}\,d\theta = \frac{\pi}{2}$. and $\int_{-1}^{1}(1-y^2)\,dy = 2-\frac{2}{3} = \frac{4}{3}$.

   Surface Area $= \int_{-1}^{1} 2\pi(2+\sqrt{1-y^2})\sqrt{\frac{1}{1-y^2}}dy + 2\pi\cdot 2\cdot 2 = 4\pi^2 + 12\pi$.

   Here we have used $\int_{-1}^{1}\frac{1}{\sqrt{1-y^2}}\,dy = \sin^{-1} y\Big|_{-1}^{1} = \pi$.

2. (12 pts) Find the solutions for (a): $\dfrac{dy}{dx} = e^{x-y},\ y(0)=1$, and (b): $x\dfrac{dy}{dx} + y = \sin x,\ y(\pi) = 0,\ x>0$, respectively.

   Ans:

   (a): $\int e^y dy = \int e^x dx$. Integration gives $e^y = e^x + C_1$ for some constant C_1.

   $y = \ln(e^x + C_1)$, $y(0)=1$, so $C_1 = e-1$,

   Answer: $y = \ln(e^x + e - 1)$.

   (b): Find the integration factor through standard procedure to obtain $(xy)' = \sin x$.

   Integration gives $y = \frac{1}{x}(-\cos x + C_2)$. Since $y(\pi)=0$, we have $C_2 = -1$.

   Answer: $y = \frac{-1}{x}(1+\cos x)$.

3. (6 pts) Write down the form of partial fraction expansion for $\dfrac{x^7}{(1-x^4)^2}$. Need NOT find the undetermined coefficients.

   Ans:

$$
\begin{aligned}
\frac{x^4}{(1-x^4)^2} &= \frac{x^7}{(1+x)^2\,(1-x)^2\,(1+x^2)^2}\\
&= \frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{1-x} + \frac{D}{(1-x)^2} + \frac{Ex+F}{1+x^2} + \frac{Gx+H}{(1+x^2)^2}.
\end{aligned}
$$

4. (64 pts) Evaluate

$$(a): \int_0^1 x^2 e^{-x} dx \quad (b): \int \cosh^4 x\,dx \quad (c): \int_0^1 \sin^{-1} x\,dx \quad (d): \int_0^{\pi/4} \tan^3 x \sec^3 x\,dx$$

$$(e): \int_1^2 \frac{1}{\sqrt{4x-x^2}}\,dx \quad (f): \int_0^1 \frac{2x+1}{(1+x^2)^2} dx \quad (g): \int_0^1 \frac{1}{\sqrt{1+e^x}}\,dx \quad (h): \int_0^{\pi/2} \frac{1}{1+\sin x} dx$$

Ans:

(a) Use integration by parts:

$$\int_0^1 x^2 e^{-x}\,dx = -x^2 e^{-x} - 2xe^{-x} - 2e^{-x}\Big|_0^1 = 2 - \frac{5}{e}$$

(b)

$$(\cosh^2 x)^2 = (\frac{\cosh 2x + 1}{2})^2 = \frac{1}{4}\cosh^2 2x + \frac{1}{2}\cosh 2x + \frac{1}{4} = \frac{1}{8}(\cosh 4x + 1) + \frac{1}{2}\cosh 2x + \frac{1}{4}$$

$$\int \cosh^4 x\,dx = \frac{1}{32}\sinh 4x + \frac{1}{4}\sinh 2x + \frac{3}{8}x + C$$

(c)

$$\int_0^1 \sin^{-1} x\,dx = x \sin^{-1} x\Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}} dx = \left(x\sin^{-1} x + \sqrt{1-x^2}\right)\Big|_0^1 = \frac{\pi}{2} - 1$$

(d) Let $u = \sec x$

$$\int_0^{\pi/4} \tan^3 x \sec^3 x\,dx = \int_1^{\sqrt{2}} (u^2-1)u^2\,du = (\frac{u^5}{5} - \frac{u^3}{3})|_1^{\sqrt{2}} = \frac{2+2\sqrt{2}}{15}$$

(e) Let $x - 2 = 2\sin\theta$

$$\int_1^2 \frac{1}{\sqrt{4x-x^2}}\,dx = \int_{x=1}^2 \frac{1}{\sqrt{4-(x-2)^2}}\,dx = \int_{\theta=-\frac{\pi}{6}}^0 \frac{1}{2\cos\theta} 2\cos\theta\,d\theta = \int_{-\frac{\pi}{6}}^0 1\,d\theta = \frac{\pi}{6}$$

(f)

$$\int_0^1 \frac{2x+1}{(1+x^2)^2} dx = \int_0^1 \frac{2x}{(1+x^2)^2} dx + \int_0^1 \frac{1}{(1+x^2)^2} dx$$

$$\int_0^1 \frac{2x}{(1+x^2)^2} dx = \int_0^1 \frac{d(1+x^2)}{(1+x^2)^2} = -(1+x^2)^{-1}\Big|_0^1 = \frac{1}{2}$$

$$\int_0^1 \frac{1}{(1+x^2)^2} dx \overset{x=\tan t}{=} \int_{t=0}^{\frac{\pi}{4}} \cos^2 t\,dt = \int_0^{\frac{\pi}{4}} \frac{\cos 2t + 1}{2}\,dt = \frac{\pi}{8} + \frac{1}{4}$$

Answer $= \dfrac{\pi}{8} + \dfrac{3}{4}$

(g) With the substitution $u = \sqrt{1+e^x}$, $2u\,du = e^x\,dx$, $e^x = u^2 - 1$,

$$\int_0^1 \frac{1}{\sqrt{1+e^x}}\,dx = \int_{\sqrt{2}}^{\sqrt{1+e}} \frac{1}{u}\frac{2u}{u^2-1}\,du = \int_{\sqrt{2}}^{\sqrt{1+e}} \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du$$

$$= \ln\left|\frac{u-1}{u+1}\right| \Big|_{\sqrt{2}}^{\sqrt{1+e}} = \ln\left|\frac{\sqrt{e+1}-1}{\sqrt{e+1}+1}\right| - \ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right| = 2\ln\left(\frac{\sqrt{e+1}-1}{\sqrt{2}-1}\right) - 1$$

(h) Method 1:

With the substitution $t = \tan\frac{x}{2}$ $(x = 2\tan^{-1} t)$, $\sin x = \frac{2t}{1+t^2}$, $dx = \frac{2\,dt}{1+t^2}$,

$$\int_{x=0}^{\pi/2} \frac{1}{1+\frac{2t}{1+t^2}} \frac{2dt}{1+t^2} = \int_{t=0}^{1} \frac{2dt}{(1+t)^2} = -2(1+t)^{-1}\Big|_0^1 = 1.$$

Method 2:

$$\int_{x=0}^{\pi/2} \frac{1}{1+\sin x}dx = \int_{x=0}^{\pi/2} \frac{1-\sin x}{(1-\sin x)(1+\sin x)}dx = \int_{x=0}^{\pi/2} \frac{1-\sin x}{\cos^2 x}dx$$

$$= \int_{x=0}^{\pi/2} sec^2 x - \tan x \sec x\; dx = \tan x - \sec x\Big|_{x=0}^{\pi/2} (\mathbf{4pts})$$

$$\tan x - \sec x\Big|_{x=0} = 0 - 1 = -1.$$

$$\tan x - \sec x\Big|_{\pi/2} = \frac{\sin x - 1}{\cos x}\Big|_{x\to\pi/2} \text{ (apply L'Hôpital's Rule for } \frac{0}{0}) = \frac{\cos x}{\sin x}\Big|_{x\to\pi/2} = 0$$

Answer $= 0 - (-1) = 1$. (**4 pts**)

5. (8 pts) Order e^x, x^x, $(\ln x)^x$ and x^e from slowest to fastest growing rate as $x \to \infty$. Explain.

Ans:

$$\lim_{x\to\infty} \frac{x^e}{e^x} = \lim_{x\to\infty} \frac{ex^{e-1}}{e^x}i = \lim_{x\to\infty} \frac{e(e-1)x^{e-2}}{e^x}i = \lim_{x\to\infty} \frac{e(e-1)(e-2)x^{e-3}}{e^x} = 0\ (\because e < 3).$$

$$\lim_{x\to\infty} \frac{e^x}{(\ln x)^x} = \lim_{x\to\infty} \frac{e^x}{e^{x\ln\ln x}} = \lim_{x\to\infty} \frac{1}{e^{x(\ln\ln x - 1)}} = 0 \text{ (since } \ln\ln x > 1 \text{ for } x \text{ large enough).}$$

$$\lim_{x\to\infty} \frac{(\ln x)^x}{x^x} = \lim_{x\to\infty} \frac{e^{x\ln\ln x}}{e^{x\ln x}} = \lim_{x\to\infty} \frac{1}{e^{x(\ln x - \ln\ln x)}} = 0\ (\because \ln x > \ln\ln x \text{ for } x \text{ large enough).}$$

Answer: From slowest to fastest: $x^e \ll e^x \ll (\ln x)^x \ll x^x$.

6. (10 pts) let f^{-1} be the inverse function of f. Suppose f and f^{-1} are twice differentiable with

$$f(1) = 2, \quad f'(1) = 3, \quad f''(1) = 4,$$
$$f(2) = 5, \quad f'(2) = 6, \quad f''(2) = 7,$$

$$f(3) = 8, \quad f'(3) = 9, \quad f''(3) = 10,$$

Find $(f^{-1})''(2)$.

Ans:

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

$$\frac{d}{dy} \implies (f^{-1})''(y) = \frac{-f''\big(f^{-1}(y)\big)}{\Big(f'\big(f^{-1}(y)\big)\Big)^3},$$

$$f(1) = 2 \implies f^{-1}(2) = 1.$$

$$(f^{-1})''(2) = \frac{-f''\big(f^{-1}(2)\big)}{\Big(f'\big(f^{-1}(2)\big)\Big)^3} = \frac{-f''\big(1\big)}{\Big(f'\big(1\big)\Big)^3} = \frac{-4}{27}$$

7. (10 pts) State both parts of Fundamental Theorem of Calculus. Then prove that 'part 1 implies part 2'.

Ans:

See the textbook or Lecture 19 note.